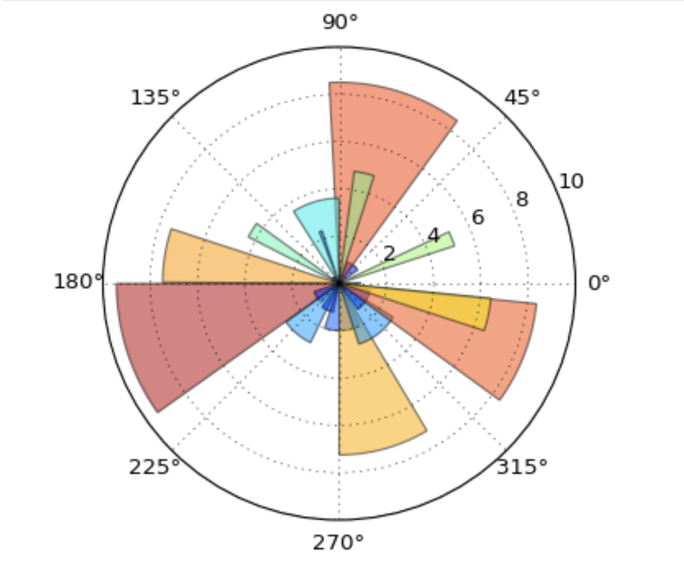
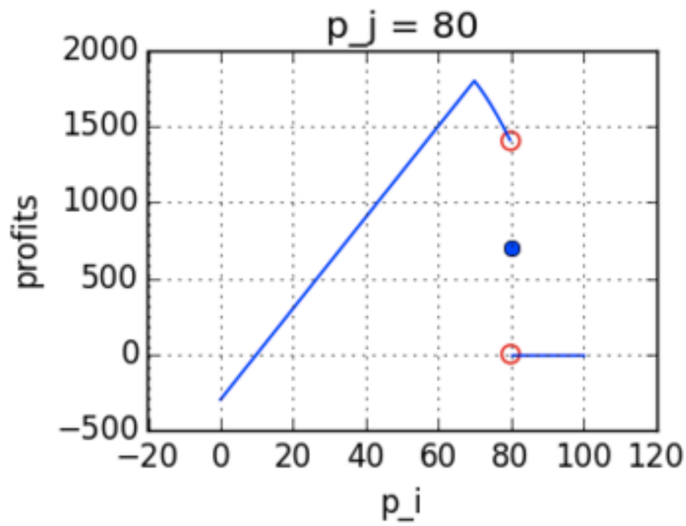
Solution Manual

DSE & ISI MA (Economics) Entrances





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1. ISI - 2013

Solved Problems - ISI 2013 - PEB

Exercise 1.1 An agent earns w units of wage while young, and earns nothing while old. The agent lives for two periods and consumes in both the periods. The utility function for the agent is given by $u = \log c_1 + \log c_2$, where c_i is the consumption in period $i = 1, 2$. The agent faces a constant rate of interest r (net interest rate) at which it can freely lend or borrow,

(a) Find out the level of saving of the agent while young.

(b) What would be the consequence of a rise in the interest rate, r, on the savings of the agent?

A 1.1

Agent's utility maximization problem is the following:

$$\begin{aligned} \max_{c_1, c_2} \quad & \log c_1 + \log c_2 \\ \text{s.t.} \quad & c_1(1+r) + c_2 = w(1+r) \\ \& \quad & c_1 \geq 0,\ c_2 \geq 0 \end{aligned}$$

(a) Solving the above problem we get:
$$(c_1, c_2) = \left(\frac{w}{2}, \frac{w(1+r)}{2}\right)$$
Hence, Savings $= w - c_1 = \dfrac{w}{2}$

(b) Clearly, Savings doesn't change with change in rate of interest rate, r.

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Exercise 1.2 Consider a city that has a number of fast food stalls selling Masala Dosa (MD). All vendors have a marginal cost of Rs. 15 per MD, and can sell at most 100 MD a day.

(a) If the price of a MD is Rs. 20, how much does each vendor want to sell?
(b) If demand for MD be $d(p) = 4400 - 120p$, where p denotes price per MD, and each vendor sells exactly 100 units of MD, then how many vendors selling MD are there in the market?
(c) Suppose that the city authorities decide to restrict the number of vendors to 20. What would be the market price of MD in that case?
(d) If the city authorities decide to issue permits to the vendors keeping the number unchanged at 20, what is the maximum that a vendor will be willing to pay for obtaining such a permit?

A 1.2

(a) If the price of a MD is Rs. 20 and the marginal cost is Rs. 15 per MD, vendor's profit maximization problem is the following:

$$\max_m \quad 20m - 15m$$
$$\text{s.t.} \quad 0 \leq m \leq 100$$

Thus, each vendor would want to sell 100 MD a day.

(b) Given competitive behavior, free entry-exit from the industry and constant returns to scale technology, we have zero profit condition, that is, price equals marginal cost. Thus, demand is

$$d(15) = 4400 - 120(15) = 2600$$

Since each vendor sells 100 units and demand is 2600 units, there are 26 vendors selling MD in the market.

(c) If number of vendors are 20 and each vendor produces 100 units, price is given by

$$p = \frac{4400 - 2000}{120} = 20$$

(d) The maximum price that a vendor is willing to pay for the permit is equal to the profit that a vendor gets if he operates i.e. $5 \times 100 = 500$.

Exercise 1.3 A firm is deciding whether to hire a worker for a day at a daily wage of Rs. 20. If hired, the worker can work for a maximum of 10 hours during the day. The worker can be used to produce two intermediate inputs, 1 and 2, which can then be combined to produce a final good. If the worker produces only 1, then he can produce 10 units of input 1 in an hour. However, if the worker produces only 2, then he can produce 20 units of input 2 in an hour. Denoting the levels of production of the amount produced of the intermediate goods by k_1 and k_2, the production function of the final good is given by $\sqrt{k_1 k_2}$. Let the final product be sold at the end of the day at a per unit price of Rs. 1. Solve for the firms optimal hiring, production and sale decision.

A 1.3

The production possibility frontier of the two inputs is given by

$$\frac{k_1}{10} + \frac{k_2}{20} = 10$$

Since final product can be sold at the end of the day at a per unit price of Rs. 1. the firm's profit maximization problem is

$$\begin{aligned} \max_{k_1,k_2} \quad & \sqrt{k_1 k_2} - 20 \\ \text{s.t.} \quad & \frac{k_1}{10} + \frac{k_2}{20} = 10 \\ \& \quad & k_1 \geq 0,\ k_2 \geq 0, \end{aligned}$$

Thus, firm will hire the worker, produces $k_1 = 50$, $k_2 = 100$ and produces and sells output $= 50\sqrt{2}$.

Exercise 1.4 A monopolist has contracted with the government to sell as much of its output as it likes to the government at Rs. 100 per unit. Its sales to the government are positive, and it also sells its output to buyers at Rs. 150 per unit. What is the price elasticity of demand for the monopolist's services in the private market?

A 1.4

Since monopolist's sale to the government is positive, his marginal revenue at the point of sale in the private market must be Rs. 100. Now price in the private market is Rs. 150. We can compute the price elasticity of demand in the following way:

$$\text{TR}(x) = p(x) \cdot x$$

Differentiating $\text{TR}(x)$ w.r.t. x, we get,

$$\begin{aligned} \text{MR}(x) &= p(x) + x\frac{dp(x)}{dx} \\ \text{MR}(x) &= p(x) + p(x)\frac{x}{p(x)}\frac{dp(x)}{dx} \\ \text{MR}(x) &= p(x)\left(1 + \frac{1}{\eta}\right) \end{aligned}$$

Now substituting $p(x) = 150$ and $\text{MR}(x) = 100$ in above we get elasticity, $\eta = -3$

Exercise 1.5 Consider the following production function with usual notations.

$$Y = K^{\alpha}L^{1-\alpha} - \beta K + \theta L \text{ with } 0 < \alpha < 1, \beta > 0, \theta > 0$$

Examine the validity of the following statements.

(a) Production function satisfies constant returns to scale.
(b) The demand function for labour is defined for all non-negative wage rates.
(c) The demand function for capital is undefined when price of capital service is zero.

A 1.5

(a) Let $f(K,L)$ denotes the production function.

$$\begin{aligned} f(tK,tL) &= (tK)^{\alpha}(tL)^{1-\alpha} - \beta tK + \theta tL \\ &= t(K^{\alpha}L^{1-\alpha} - \beta K + \theta L) \\ &= tf(K,L) \end{aligned}$$

Thus, production function satisfies constant returns to scale.

(b) Profit maximization problem of the competitive producer is

$$\begin{aligned} \max_{K,L} \quad & K^{\alpha}L^{1-\alpha} - \beta K + \theta L - wL - rK \\ \text{s.t.} \quad & L \geq 0,\ K \geq 0 \end{aligned}$$

(b) The above problem is equivalent to

$$\begin{aligned} \max_{K,L} \quad & K^{\alpha}L^{1-\alpha} - (r+\beta)K - (w-\theta)L \\ \text{s.t.} \quad & L \geq 0,\ K \geq 0 \end{aligned}$$

Clearly, when $0 \leq w \leq \theta$, demand function for labor is not defined.

(c) Also, the demand function for capital is defined when price of capital service is zero provided $w > \theta$.

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Exercise 1.6 Suppose that due to technological progress labour requirement per unit of output is halved in a Simple Keynesian model where output is proportional to the level of employment. What happens to the equilibrium level of output and the equilibrium level of employment in this case? Consider a modified Keynesian model where consumption expenditure is proportional to labour income and wage-rate is given. Does technological progress produce a different effect on the equilibrium level of output in this case?

A 1.6

Suppose the production function has changed from $Y = F_0(L) = aL$ to $Y = F_1(L) = F_0(2L) = 2aL$ where $a > 0$. Labor demand curve is, therefore,

$$L_0(w) \in \begin{cases} \{0\} & \text{if } a < w \\ \mathbb{R}_+ & \text{if } a = w \\ \emptyset & \text{if } a > w \end{cases}$$

where w is the real wage and it changes to

$$L_1(w) \in \begin{cases} \{0\} & \text{if } 2a < w \\ \mathbb{R}_+ & \text{if } 2a = w \\ \emptyset & \text{if } 2a > w \end{cases}$$

Let us assume that Labor supply is exogenously given and is equal to $\bar{L}$. Solving for the equilibrium in labor market we get that the real wage has changed from a to $2a$ but the equilibrium employment is $\bar{L}$ in both cases. Therefore, Aggregate Supply curve has shifted from $Y = a\bar{L}$ to $Y = 2a\bar{L}$. Given any aggregate demand curve, $Y = \text{AD}(w)$ it is easy to see that the new equilibrium level of output in the model will be twice as much as it was earlier.

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Exercise 1.7 A positive investment multiplier does not exist in an open economy simple Keynesian model when the entire amount of investment goods is supplied from import. Examine the validity of this statement.

A 1.7

In the simple Keynesian Model, we consider a static set up where Goods market equilibrium condition in the open economy is

$$Y = C(Y) + I + G + X - M$$

If the entire investment good is supplied from import the net demand is 0 ad therefore the multiplier is 0. Therefore, a positive investment multiplier does not exist in an open economy simple Keynesian model when the entire amount of investment goods is supplied from import.

Exercise 1.8 A consumer consumes two goods, x_1 and x_2, with the following utility function

$$U(x_1, x_2) = u(x_1) + u(x_2)$$

Suppose that the income elasticity is positive. It is claimed that in the above set-up all goods are normal. Prove or disprove this claim.

A 1.8

It is trivial. Given that income elasticity of demand is positive, goods must be normal.

Exercise 1.9 A consumer derives his market demand, say x, for the product X as $x = 10 + \dfrac{m}{10p_x}$,where $m > 0$ is his money income and p_x is the price per unit of X. Suppose that initially he has money income $m = 120$, and the price of the product is $p_x = 3$. Further, the price of the product is now changed to $p'_x = 2$. Find the price effect. Then decompose price effect into substitution effect and income effect.

A 1.9

Given that the money income is $m = 120$, demand is $x = 10 + \dfrac{12}{p_x}$. Demand for X when price is $p_x = 3$ is 14. Demand for X when price is $p'_x = 2$ is 16. In order to find the Substitution effect and Income effect, we first need to find that what must be his income so that he can afford the original equilibrium at the new price. Original equilibrium was $(14, 78)$ where 78 is his remaining income after spending on X $(= 120 - 14(3))$. At the new price we need to give him income $m' = 14(2) + 78 = 106$ so that he can exactly afford his original consumption bundle at the new prices. We will now find the demand for X in this situation when $m' = 106$ and price is 2, and we get $x = 10 + \dfrac{106}{20} = 15.3$. Therefore, Substitution effect is $15.3 - 14 = 1.3$ and income effect is $16 - 15.3 = 0.7$.

Exercise 1.10 Consider an otherwise identical Solow model of economic growth where the entire income is consumed.

(a) Analyse how wage and rental rate on capital would change over time.
(b) Can the economy attain steady state equilibrium?

A 1.10

(a) In the Solow model, where the entire income is consumed, there will be no capital formation and in the presence of depreciation capital depletes over time. If the population is either fixed or grows over time then there will be fall in capital by labor ratio over time. Therefore wage rate would fall and rental rate would increase over time.
(b) Yes, this economy will attain a steady state at $k = 0$ which is disappointing but steady.

2. ISI - 2015

Solved Problems - ISI 2015 - PEB

Exercise 2.1 Consider an agent in an economy with two goods X_1 and X_2. Suppose she has income 20. Suppose also that when she consumes amounts x_1 and x_2 of the two goods respectively, she gets utility

$$u(x_1, x_2) = 2x_1 + 32x_2 - 3x_2^2$$

(a) Suppose the prices of X_1 and X_2 are each 1. What is the agent's optimal consumption bundle?
(b) Suppose the price of X_2 increases to 4, all else remaining the same. Which consumption bundle does the agent choose now?
(c) How much extra income must the agent be given to compensate her for the increase in price of X_2?

A 2.1

(a) Check yourself that $u(x_1, x_2)$ is a concave function and hence is also quasi-concave, therefore solution to the above problem can be obtained through the standard slope analysis. $\text{MRS}_{12} = \dfrac{2}{32 - 6x_2}$ and the budget line is $x_1 + x_2 = 20$. Solution is not at the corner $x_2 = 0$ because at this corner, $\text{MRS}_{12} = \dfrac{1}{16} < 1 = \dfrac{p_1}{p_2}$, therefore consumer will benefit from spending some money on x_2. Solution is not at the other corner $x_2 = 20$ because $\text{MU}_2 < 0$ at this consumption level, therefore it is beneficial to spend less on x_2. Hence, the solution is in the interior and satisfy $\text{MRS}_{12} = 1$, and we get $x_2 = 5$ and $x_1 = 15$.

A 2.1

(b) For the new problem, solution is not at the corner $x_2 = 0$ because $\text{MRS}_{12} = \frac{1}{16} < \frac{1}{4} = \frac{p_1}{p_2}$, therefore it pays to move some money to x_2. Solution is not at the other corner $x_2 = 20$ because $\text{MU}_2 < 0$ at this consumption level, therefore it pays to move money to x_1. Therefore the solution satisfy $\text{MRS}_{12} = \frac{1}{4}$, and we get $x_2 = 4$ and $x_1 = 4$.

(c) In order to find the extra income, we just need to find the x_1 at which the individual will attain the same level of satisfaction as in (a) at the prices specified in part (b). Given that the price ratio is $\frac{1}{4}$, $x_2 = 4$. To find x_1, we will solve the following for x_1: $2x_1 + 32(4) - 3(4^2) = 2(15) + 32(5) - 3(5^2)$ (we equate the satisfaction level in (a) to the satisfaction level from a bundle in which $x_2 = 4$). Thus, $x_1 = 17.5$. Now we find the income needed to afford $x_1 = 17.5$ and $x_2 = 4$ at prices $(1,4)$ and we get 33.5 which is 13.5 higher than his current income. So, the compensation needed is 13.5.

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Exercise 2.2 Suppose a government agency has a monopoly in the provision of internet connections. The marginal cost of providing internet connections is 1/2, whereas the inverse demand function is given by: $p = 1 - q$. The official charge per connection is set at 0; thus, the state provides a subsidy of 1/2 per connection. However, the state can only provide budgetary support for the supply of 0.4 units, which it raises through taxes on consumers. Bureaucrats in charge of sanctioning internet connections are in a position to ask for bribes, and consumers are willing to pay them in order to get connections. Bureaucrats cannot, however, increase supply beyond 0.4 units.

(a) Find the equilibrium bribe rate per connection and the social surplus.

(b) Now suppose the government agency is privatized and the market is deregulated; however, due large fixed costs of entry relative to demand, the privatized company continues to maintain its monopoly. Find the new equilibrium price, bribe rate and social surplus, specifying whether privatization increases or reduces them.

(c) Suppose now a technological innovation becomes available to the privatized monopoly, which reduces its marginal cost of providing an internet connection to c, $0 < c < 1/2$. Find the range of values of c for which privatization increases consumers' surplus.

A 2.2

(a) Equilibrium bribe rate per connection will be 0.6 and the net social surplus will be the consumer surplus on consumption of 0.4 units *plus* bribe *minus* the marginal cost that equals $0.08 + (0.6 - 0.5)0.4 = 0.12$.

(b) With privatization, equilbrium quantity will be $1/4$ and equilibrium price will be $3/4$. Social surplus, monopoly profilts *plus* consumer surplus, is equal to $3/32$.

(c) For $0 < c < 1/2$, monopoly quantity is $(1-c)/2$ and monopoly price is $(1+c)/2$. Consumer Surplus (in case of monopoly) as a function of c equals $(1-c)^2/8$. Range of values of c for which privatization increases consumers' surplus satisfy $(1-c)^2/8 > 0.08$ that gives us $0 < c < 0.2$.

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Exercise 2.3 Suppose the borders of a state, B, coincide with the circumference of a circle of radius $r > 0$, and its population is distributed uniformly within its borders (so that the proportion of the population living within some region of B is simply the proportion of the state's total land mass contained in that region), with total population normalized to 1. For any resident of B, the cost of travelling a distance d is kd, with $k > 0$. Every resident of B is endowed with an income of 10, and is willing to spend up to this amount to consume one unit of a good, G, which is imported from outside the state at zero transport cost. The Finance Minister of B has imposed an entry tax at the rate $100t\%$ on shipments of G brought into B. Thus, a unit of G costs $p(1+t)$ inside the borders of B, but can be purchased for just p outside; $p(1+t) < 10$. Individual residents of B have to decide whether to travel beyond its borders to consume the good or to purchase it inside the state. Individuals can travel anywhere to shop and consume, but have to return to their place of origin afterwards.

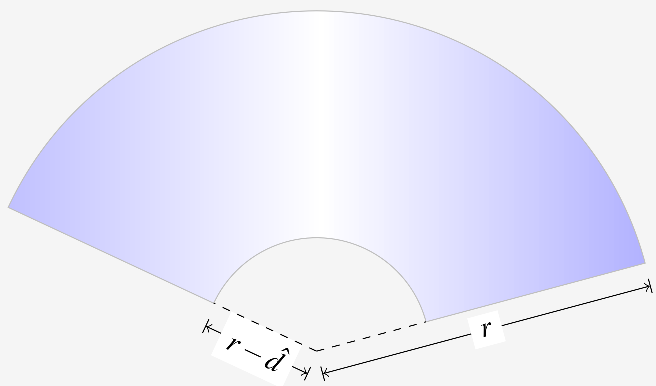
(a) Find the proportion of the population of B which will evade the entry tax by shopping outside the state.

(b) Find the social welfare-maximizing tax rate. Also find the necessary and sufficient conditions for it to be identical to the revenue-maximizing tax rate.

(c) Assume that the revenue-maximizing tax rate is initially positive. Find the elasticity of tax revenue with respect to the external price of G, supposing that the Finance Minister always chooses the revenue-maximizing tax rate.

A 2.3

(a) An individual located at distance $r - \hat{d}$ from the center of B will be indifferent between buying G from inside and outside if $p(1+t) = p + 2k\hat{d}$ or equivalently, consumer located $\hat{d} = \frac{pt}{2k}$ distance inside the boundary is indifferent between buying from inside and outside the region. Therefore, proportion of people who will buy from outside the circular region equals $1 - \frac{\pi(r-\hat{d})^2}{\pi r^2} = 1 - \frac{(r-\hat{d})^2}{r^2}$.



A 2.3

(b) Given the tax rate t, social welfare of B is given by the sum of welfare of people buying from outside the state *plus* the welfare of the people buying G from inside the state and the Government tax revenue.

$$\int_{x=0}^{\hat{d}} \frac{2\pi(r-x)}{\pi r^2}(10-p-2kx)dx + \frac{\pi(r-\hat{d})^2}{\pi r^2}(10-p) \quad \text{where } \hat{d} = \frac{pt}{2k}$$

Differentiating it with respect to t, we get the first order condition

$$\frac{2\pi(r-\hat{d})}{\pi r^2}(10-p-2k\hat{d})\frac{p}{2k} - \frac{2\pi(r-\hat{d})}{\pi r^2}(10-p)\frac{p}{2k} = 0 \quad \text{where } \hat{d} = \frac{pt}{2k}$$

The above holds when

$\hat{d} = 0$ (welfare maximizing) or $\hat{d} = r$ (welfare minimizing)

Therefore, the social welfare maximizing tax rate is $t = k\hat{d}/p = 0$.

Now to find tax revenue maximizing tax rate, we will first write the expression for Tax-Revenue:

$$pt\frac{\pi(r-\hat{d})^2}{\pi r^2} = 2k\hat{d}\frac{\pi(r-\hat{d})^2}{\pi r^2} \quad \left[\text{because } \hat{d} = \frac{pt}{2k}\right]$$

Finding the revenue maximizing t is equivalent to finding the revenue maximizing $\hat{d}$. Maximizing above, we get

$$\hat{d} = \frac{r}{3}$$

and hence the revenue maximizing tax rate

$$t = \frac{2rk}{3p}$$

Therefore, necessary and sufficient condition for the the revenue maximizing tax rate to be the same as welfare maximizing tax rate is $k = 0$.

(c) From (b), we know that revenue maximizing tax rate is

$$t = \frac{2rk}{3p}$$

and the revenue is

$$pt\frac{(r-[pt/(2k)])^2}{r^2}$$

Therefore, optimal revenue is $\frac{4k}{9}$

Therefore, the elasticity of tax revenue with respect to p is 0.

■

Exercise 2.4 Suppose there are two firms, 1 and 2, each producing chocolate, at 0 marginal cost. However, one firm's product is not identical to the product of the other. The inverse demand functions are as follows:

$$p_1 = A_1 - b_{11}q_1 - b_{12}q_2, \; p_2 = A_2 - b_{21}q_1 - b_{22}q_2;$$

where p_1 and q_1 are respectively price obtained and quantity produced by firm 1 and p_2 and q_2 are respectively price obtained and quantity produced by firm 2. $A_1, A_2, b_{11}, b_{12}, b_{21}, b_{22}$ are all positive. Assume the firms choose independently how much to produce.

(a) How much do the two firms produce, assuming both produce positive quantities?

(b) What conditions on the parameters $A_1, A_2, b_{11}, b_{12}, b_{21}, b_{22}$ are together both necessary and sufficient to ensure that both firms produce positive quantities?

(c) Under what set of conditions on these parameters does this model reduce to the standard Cournot model?

A 2.4

(a) Given the inverse demand functions:

$p_1 = A_1 - b_{11}q_1 - b_{12}q_2, \; p_2 = A_2 - b_{21}q_1 - b_{22}q_2;$

We can solve for the best response function of the firm 1 by maximizing:

$\max_{q_1} \quad A_1q_1 - b_{11}q_1^2 - b_{12}q_2q_1$

and of firm 2 by maximizing:

$\max_{q_2} \quad A_2q_2 - b_{21}q_1q_2 - b_{22}q_2^2$

So we get

$$\begin{aligned} \mathrm{BR}_1(q_2) &= \max\left\{\frac{A_1 - b_{12}q_2}{2b_{11}}, 0\right\} \\ \mathrm{BR}_2(q_1) &= \max\left\{\frac{A_2 - b_{21}q_1}{2b_{22}}, 0\right\} \end{aligned}$$

If both firms produce positive quantities then the output of the two firms solves

$$\begin{aligned} q_1 &= \frac{A_1 - b_{12}q_2}{2b_{11}} \\ q_2 &= \frac{A_2 - b_{21}q_1}{2b_{22}} \end{aligned}$$

We solve the above system to get

$$q_1^* = \frac{2A_1b_{22} - A_2b_{12}}{4b_{11}b_{22} - b_{21}b_{12}}; \qquad q_2^* = \frac{2A_2b_{11} - A_1b_{21}}{4b_{11}b_{22} - b_{21}b_{12}}$$

(b) $q_1^* > 0$ and $q_2^* > 0$ in equilibrium *if and only if* $\frac{2A_1b_{22} - A_2b_{12}}{4b_{11}b_{22} - b_{21}b_{12}} > 0$ and $\frac{2A_2b_{11} - A_1b_{21}}{4b_{11}b_{22} - b_{21}b_{12}} > 0.$

(c) When $b_{11} = b_{12} = b_{21} = b_{22}$ and $A_1 = A_2$, then this model reduces to the standard Cournot model.

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Exercise 2.5 Suppose a firm manufactures a good with labor as the only input. Its production function is $Q = L$, where Q is output and L is total labor input employed. Suppose further that the firm is a monopolist in the product market and a monopsonist in the labor market. Workers may be male (M) or female (F); thus, $L = L_M + L_F$.Let the inverse demand function for output and the supply functions for gender-specific labor be respectively $p = A - \frac{q}{2}$; $L_i = e_i^{\varepsilon_i}$ where p is the price received per unit of the good and ED is the wage the firm pays to each unit of labor of gender i; $i \in \{M, F\}$. Let $\varepsilon_M\varepsilon_F = 1$. Suppose, in equilibrium, the firm is observed to hire both M and F workers, but pay M workers double the wage rate that it pays F workers.

(a) Derive the exact numerical value of the elasticity of supply of male labor.

(b) What happens to total male labor income as a proportion of total labor income when the output demand parameter A increases? Prove your claim.

A 2.5

Firm solves the following profit maximisation problem:

$$\begin{aligned} \max \quad & pQ - w_F L_F - w_M L_M \\ \text{s.t.} \quad & Q = L_F + L_M \\ & p = A - \frac{Q}{2} \\ & L_F = w_F^{\varepsilon_F} \\ & L_M = w_M^{\varepsilon_M} \end{aligned}$$

The firm chooses $(p, Q, w_F, L_F, w_M, L_M)$ since it is a monopolist as well as a monopsonist. By eliminating Q and P using the demand constraints and the production function, the above problem can be rewritten as:

$$\begin{aligned} \max \quad & \left(A - \frac{L_F + L_M}{2}\right)(L_F + L_M) - w_F L_F - w_M L_M \\ \text{s.t.} \quad & L_F = w_F^{\varepsilon_F} \\ & L_M = w_M^{\varepsilon_M} \end{aligned}$$

Now we will use the labor supply equations to write the above problem just in terms of input prices:

$$\max \quad \left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)(w_F^{\varepsilon_F} + w_M^{\varepsilon_M}) - w_F^{1+\varepsilon_F} - w_M^{1+\varepsilon_M}$$

FOCs:

$$\left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)\varepsilon_F w_F^{\varepsilon_F - 1} - \frac{1}{2}(w_F^{\varepsilon_F} + w_M^{\varepsilon_M})\varepsilon_F w_F^{\varepsilon_F - 1} - (1+\varepsilon_F)w_F^{\varepsilon_F} = 0$$

$$\left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)\varepsilon_M w_M^{\varepsilon_M - 1} - \frac{1}{2}(w_F^{\varepsilon_F} + w_M^{\varepsilon_M})\varepsilon_M w_M^{\varepsilon_M - 1} - (1+\varepsilon_M)w_M^{\varepsilon_M} = 0$$

Above can be rewritten as

$$\left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)\varepsilon_F - \frac{1}{2}(w_F^{\varepsilon_F} + w_M^{\varepsilon_M})\varepsilon_F - (1+\varepsilon_F)w_F = 0$$

$$\left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)\varepsilon_M - \frac{1}{2}(w_F^{\varepsilon_F} + w_M^{\varepsilon_M})\varepsilon_M - (1+\varepsilon_M)w_M = 0$$

Again, above can be rewritten as

$$\left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)\varepsilon_F - \frac{1}{2}(w_F^{\varepsilon_F} + w_M^{\varepsilon_M})\varepsilon_F = (1+\varepsilon_F)w_F$$

$$\left(A - \frac{w_F^{\varepsilon_F} + w_M^{\varepsilon_M}}{2}\right)\varepsilon_M - \frac{1}{2}(w_F^{\varepsilon_F} + w_M^{\varepsilon_M})\varepsilon_M = (1+\varepsilon_M)w_M$$

A 2.5

Dividing them we get

$$\frac{\varepsilon_F}{\varepsilon_M} = \frac{(1+\varepsilon_F)w_F}{(1+\varepsilon_M)w_M}$$

Using $\varepsilon_M \varepsilon_F = 1$ and $w_M = 2w_F$, we get $\varepsilon_M = 2, \varepsilon_F = \frac{1}{2}$

Exercise 2.6 An economy comprises of a consolidated household sector, a firm sector and the government. The household supplies labour (L) to the firm. The firm produces a single good (L) by means of a production function $Y = F(L); F' > 0, F'' < 0$, and maximizes profits $\Pi = PY - WL$, where P is the price of Y and W is the wage rate.The household, besides earning wages, is also entitled to the profits of the firm. The household maximizes utility (U), given by:

$$U = \frac{1}{2}\ln C + \frac{1}{2}\ln \frac{M}{P} - d(L)$$

where C is consumption of the good and $\frac{M}{P}$ is real balance holding. The term $d(L)$ denotes the disutility from supplying labour; with $d' > 0$, $d'' > 0$. The household's budget constraint is given by:

$$PC + M = WL + \Pi + \overline{M} - PT;$$

where $\overline{M}$ is the money holding the household begins with, M is the holding they end up with and T is the real taxes levied by the government. The government's demand for the good is given by G. The government's budget constraint is given by:

$$M - \overline{M} = PG - PT;$$

Goods market clearing implies: $Y = C + G$.

(a) Prove that $\frac{dY}{dG} \in (0,1)$, and that government expenditure crowds out private consumption (i.e., $\frac{dC}{dG} < 0$).

(b) Show that everything else remaining the same, a rise in $\overline{M}$ leads to an equiproportionate rise in P.

A 2.6

Supply of output (Y) and demand for input (L): Firm chooses (Y,L) by maximizing profit

$$\Pi = PY - WL$$

subject to the constraint

$$Y = F(L)$$

Private Demand for output (C), demand for money balances (M) and supply of input (L): Household choose (C,M,L) by maximising

$$U = \frac{1}{2}\ln C + \frac{1}{2}\ln\frac{M}{P} - d(L)$$

subject to the constraint

$$PC + M = WL + \Pi + \overline{M} - PT$$

Government's demand for the final good is given by (G), and it must satisfy the budget constraint:

$$M - \overline{M} = PG - PT$$

Private Demand for the final good + Government demand for it = Supply of the final good:

$$Y = C + G$$

Demand = Supply in input market and money market: Also, market clearing conditions in labor market and money market tells us that the equilibrium labor employment must be the solution of both the household's problem and the firm's problem and equilibrium level of money balances must solve the household' utility maximisation problem and satisfy the Government's Budget constraint.

Therefore, given the exogenous variables $(G,\overline{M})$, equilibrium of the above economy consists of prices (W,P) and the endogenous variables (Y,C,M,T,Π,L) such that the above-mentioned holds. i.e., (Y,L) solves the firm's problem given (W,P), (C,M,L) solves the household's problem given (W,P) and T must satisfy the budget constraint of the government. Finally, (W,P) must be such that demand equals supply holds in all the markets.

A 2.6

Now we will will write the conditions that the equilibrium prices (W,P) and the equilibrium vector (Y,C,M,T,Π,L) must satisfy: From firm's profit maximisation problem:

$$\begin{aligned} Y &= F(L) \\ F'(L) &= \frac{W}{P} \\ \Pi &= PY - WL \end{aligned}$$

From household's utility maximisation problem:

$$\begin{aligned} PC + M = WL + \Pi + \overline{M} - PT \\ M = PC \\ d'(L) = \frac{W}{P} \times \frac{1}{2C} \end{aligned}$$

And we have the government's budget constraint:

$$M - \overline{M} = PG - PT$$

Finally, the market clearing condition

$$Y = C + G$$

Market clearing conditions for the money market and labor market are implicit in above since we denoted labor demand and labor supply by the same variable L and money demand and money supply by the same variable M. We will reduce the above system of conditions by using the household's optimisation condition and substituting $M = PC$ everywhere in the system:

$$\begin{aligned} Y = F(L) \\ F'(L) = \frac{W}{P} \\ \Pi = PY - WL \\ PC + PC = WL + \Pi + \overline{M} - PT \\ d'(L) = \frac{W}{P} \times \frac{1}{2C} \\ PC - \overline{M} = PG - PT \\ Y = C + G \end{aligned}$$

Now we will eliminate T from the system by substituting $PT = -PC + \overline{M} + PG$ (using the government's budget constraint)

$$\begin{aligned} Y = F(L) \\ F'(L) = \frac{W}{P} \\ \Pi = PY - WL \\ PC = WL + \Pi - PG \\ d'(L) = \frac{W}{P} \times \frac{1}{2C} \\ Y = C + G \end{aligned}$$

A 2.6

Next, we will eliminate Π by substituting it with $\Pi = PY - WL$ everywhere, we will then reduce the system to

$$\begin{aligned} Y &= F(L) \\ F'(L) &= \frac{W}{P} \\ d'(L) &= \frac{W}{P} \times \frac{1}{2C} \\ Y &= C + G \end{aligned}$$

Now we eliminate $\frac{W}{P}$ by substituting $\frac{W}{P} = F'(L)$ everywhere,

$$\begin{aligned} Y &= F(L) \\ d'(L) &= F'(L) \times \frac{1}{2C} \\ Y &= C + G \end{aligned}$$

Differentiating the above system with respect to G,

$$\begin{aligned} \frac{dY}{dG} &= F'(L)\frac{dL}{dG} \\ d''(L)\frac{dL}{dG} &= \frac{F''(L)}{2C} \times \frac{dL}{dG} - \frac{F'(L)}{2C^2} \times \frac{dC}{dG} \\ \frac{dY}{dG} &= \frac{dC}{dG} + 1 \end{aligned}$$

Eliminating $\frac{dC}{dG}$, we get

$$\begin{aligned} \frac{dY}{dG} &= F'(L)\frac{dL}{dG} \\ d''(L)\frac{dL}{dG} &= \frac{F''(L)}{2C} \times \frac{dL}{dG} - \frac{F'(L)}{2C^2} \times \left(\frac{dY}{dG} - 1\right) \end{aligned}$$

Now solving for $\frac{dY}{dG}$, we get

$$\frac{dY}{dG} = \frac{(F'(L))^2}{(F'(L))^2 + 2C^2 d''(L) - F''(L)} \in (0,1)$$

The above follows from $F'(L) > 0, F''(L) < 0$ and $d''(L) > 0$. Since $\frac{dC}{dG} = \frac{dY}{dG} - 1$, we get $\frac{dC}{dG} < 0$. ■

Exercise 2.7 Consider the Solow growth model in continuous time, where the exogenous rate of technological progress, g, is zero. Consider an intensive form production function given by:

$$f(k) = k^4 - 6k^3 + 11k^2 - 6k \qquad (1)$$

where $k = \frac{K}{L}$ (the capital labour ratio).

(a) Specify the assumptions made with regard to the underlying extensive form production function $F(K,L)$ in the Solow growth model, and explain which ones among these assumptions are violated by (1).

(b) Graphically show that, with a suitable value of $(n+\delta)$ where n is the population growth rate, and $\delta \in [0,1]$ is the depreciation rate on capital, there exist three steady state equilibria.

(c) Explain which of these steady state equilibria are locally unstable, and which are locally stable. Also explain whether any of these equilibria can be globally stable.

A 2.7

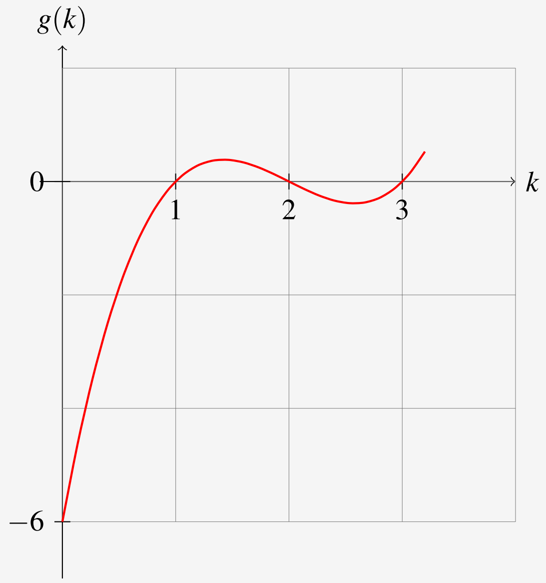
Fundamental differential equation of Solow Mdodel:

$$\dot{k} = sf(k) - (\delta + n)k$$

In the steady state $\dot{k} = 0$. If $sf(k) > (\delta + n)k$, then k increases. If $sf(k) < (\delta + n)k$, then k decreases. Let us plot $sf(k) - (\delta + n)k$ when $f(k) = k^4 - 6k^3 + 11k^2 - 6k = k(k-1)(k-2)(k-3)$. So, we will plot $k = sk(k-1)(k-2)(k-3) - (\delta + n)k$. We can easily see that one steady state is $k_0 = 0$. The other steady states (where $k \neq 0$) can be obtained by plotting

$$g(k) := \frac{\dot{k}}{k} = s(k-1)(k-2)(k-3) - (\delta + n)$$

and observing where $g(k) = 0$.



The above graph shows that there are three more steady states. For n and δ sufficiently small, these are $k_1 = 1, k_2 = 2$ and $k_3 = 3$. For $k < k_1 < g(k) < 0$ and therefore, $\dot{k} < 0$. Thus, $sf(k) < (\delta + n)k$, and k decreases. Therefore, k_1 is not locally stable. Similarly, we check for k_2 and k_3 and we will find that k_2 is locally stable and k_3 is not locally stable. Note that $k_0 = 0$ is a locally stable steady state because when k is positive and close to 0, then $\dot{k} < 0$.

Conclusion: Four steady states: k_0, k_1, k_2 and k_3. Two of them are locally stable k_0 and k_2.

■

Exercise 2.8 Consider a standard Solow model in discrete time, with the law of motion of capital is given by

$$K(t+1) = (1-\delta)K(t) + I(t)$$

where $I(t)$ is investment at time t and $K(t)$ is the capital stock at time t; the capital stock depreciates at the rate $\delta \in [0,1]$. Suppose output, $Y(t)$, is augmented by government spending, $G(t)$, in every period, and that the economy is closed; thus:

$$Y(t) = C(t) + I(t) + G(t)$$

where $C(t)$ is consumption at time t. Imagine that government spending is given by:

$$G(t) = \sigma Y(t),$$

where $\sigma \in [0,1]$.

(a) Suppose that: $C(t) = (\phi - \lambda\sigma)Y(t)$; where $\lambda \in [0,1]$ Derive the effect of higher government spending (in the form of higher σ) on the steady state equilibrium.

(b) Does a higher σ lead to a lower value of the capital stock in every period (i.e, along the entire transition path)? Prove your claim.

A 2.8

(a) In this model, the law of motion of capital is given by

$$K(t+1) = (1-\delta)K(t) + I(t)$$

Using $Y(t) = C(t) + I(t) + G(t)$, $G(t) = \sigma Y(t)$ and $C(t) = (\phi - \lambda\sigma)Y(t)$, we can rewrite the law of motion of capital as follows:

$$\begin{aligned}
K(t+1) &= (1-\delta)K(t) + Y(t) - C(t) - G(t) \\
&= (1-\delta)K(t) + Y(t) - (\phi - \lambda\sigma)Y(t) - \sigma Y(t) \\
&= (1-\delta)K(t) + (1-\phi-(1-\lambda)\sigma)Y(t) \\
&= (1-\delta)K(t) + (1-\phi-(1-\lambda)\sigma)F(K(t))
\end{aligned}$$

In the steady state, $K^* = K(t+1) = K(t)$. Therefore, K^* satisfy

$$K^* = \frac{(1-\phi-(1-\lambda)\sigma)}{\delta}F(K^*)$$

To find the effect of higher σ on K^*, we will differentiate the above expression with respect to σ:

$$\frac{dK^*}{d\sigma} = \frac{(1-\phi-(1-\lambda)\sigma)}{\delta}F'(K^*)\frac{dK^*}{d\sigma} + \frac{\lambda}{\delta}F(K^*)$$

Assuming $(1-\phi-(1-\lambda)\sigma) > \delta$, we get $\dfrac{dK^*}{d\sigma} < 0$.

A 2.8

(b) Yes, a higher σ leads to a lower value of the capital stock in every period (i.e, along the entire transition path). Let $\sigma' > \sigma$, for σ the law of motion of capital is

$$K(t+1) = (1-\delta)K(t) + (1-\phi-(1-\lambda)\sigma)F(K(t))$$

and for σ' the law of motion of capital is

$$K'(t+1) = (1-\delta)K'(t) + (1-\phi-(1-\lambda)\sigma)F(K'(t))$$

Starting from $K(0) = K'(0)$, we need to show that $K(t) > K'(t)\ \forall t \geq 1$. We will first show that $K(1) > K'(1)$,

$$\begin{aligned} K(1) &= (1-\delta)K(0) + (1-\phi-(1-\lambda)\sigma)F(K(0)) \\ &> (1-\delta)K(0) + (1-\phi-(1-\lambda)\sigma')F(K(0)) \\ &= (1-\delta)K'(0) + (1-\phi-(1-\lambda)\sigma')F(K'(0)) \\ &= K'(1) \end{aligned}$$

Suppose, by the induction procedure, $K(t) > K'(t)$ we will show that $K(t+1) > K'(t+1)$

$$\begin{aligned} K(t+1) &= (1-\delta)K(t) + (1-\phi-(1-\lambda)\sigma)F(K(t)) \\ &> (1-\delta)K(t) + (1-\phi-(1-\lambda)\sigma')F(K(t)) \\ &> (1-\delta)K'(t) + (1-\phi-(1-\lambda)\sigma')F(K'(t)) \\ &= K'(t+1) \end{aligned}$$

Therefore, higher σ lead to a lower value of the capital stock in every period.

3. DSE - 2013

Solved Problems - DSE 2013

Exercise 3.1 Starting from a stationary position, Sonia ran 100 meters in 20 seconds. Assuming that her distance from the starting point is a continuous and differentiable function $f(t)$ of time, she would definitely have run at a speed of y meters per second at some point of time, where y equals

(a) 4
(b) 20
(c) 6
(d) None of the above necessarily holds

A 3.1

(d) None of the above necessarily holds

The example $f(t) = 5t$ rules out options (a), (b) and (c).

Exercise 3.2 Suppose $f : \mathbb{R} \to \mathbb{R}$ (i.e. it is a real-valued function defined on the set of real numbers). If f is differentiable, $f(3) = 2$, and $3 \leq f'(x) \leq 4$, for all x, then it must be that $f(5)$ lies in the following interval.

(a) $[8, 10]$
(b) $[0, 8)$
(c) $(10, \infty)$
(d) $[3, 4]$

A 3.2

(a) $[8, 10]$

Given that $f(3) = 2$, and $3 \leq f'(x) \leq 4$, for all x, we have

$$f(3) + \int_3^y 3dx \leq f(y) = f(3) + \int_3^y f'(x)dx \leq f(3) + \int_3^y 4dx \quad \forall y \geq 3$$

Hence, for $y = 5$

$$\begin{aligned} & f(3) + \int_3^5 3dx \leq f(5) \leq f(3) + \int_3^5 4dx \\ \Rightarrow \quad & 2 + [3(5-3)] \leq f(5) \leq 2 + [4(5-3)] \\ \Rightarrow \quad & 8 \leq f(5) \leq 10 \end{aligned}$$

Therefore, $f(5)$ lies in the interval $[8, 10]$.

■

Exercise 3.3 The function defined by $f(x) = x(x-10)(x-20)(x-30)$ has critical points (i.e., points where $f'(x) = 0$)

(a) at some $x < 0$ and some $x > 30$.
(b) at an $x < 0$, and at some x between 0 and 30.
(c) between 0 and 10, 10 and 20, 20 and 30.
(d) None of the above captures all the critical points.

A 3.3

(c) between 0 and 10, 10 and 20, 20 and 30

Differentiating f, we get

$$\begin{aligned} f'(x) &= x(x-10)(x-20) + x(x-10)(x-30) + \\ & \quad x(x-20)(x-30) + (x-10)(x-20)(x-30) \end{aligned}$$

Since $f'(x)$ is continuous and

$$\begin{aligned} f'(0) &= -6000 \\ f'(10) &= 6000 \\ f'(20) &= -6000 \\ f'(30) &= 6000 \end{aligned}$$

therefore, by Intermediate Value theorem, $f'(x) = 0$ occurs at some point between 0 and 10, 10 and 20, 20 and 30.

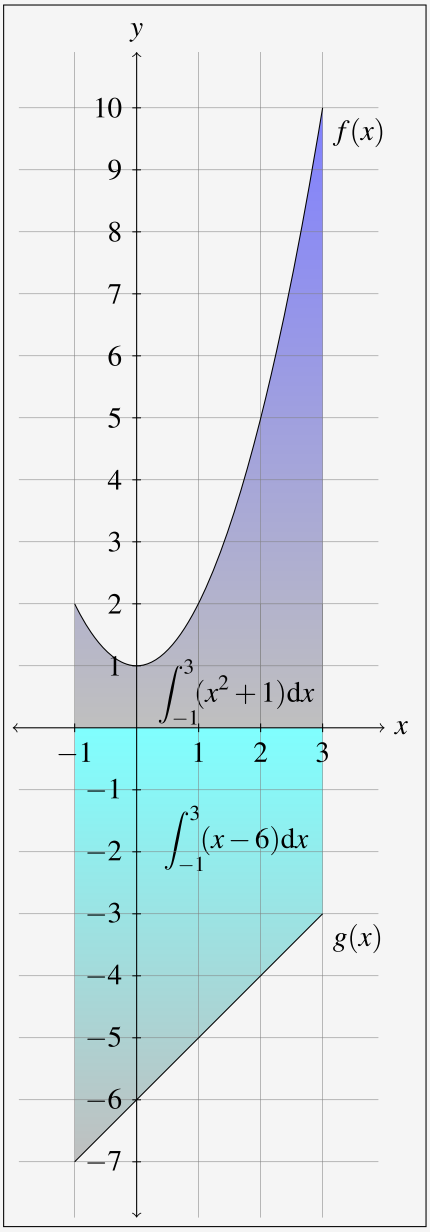
■

Exercise 3.4 The area of the region bounded above by $f(x) = x^2 + 1$ and below by $g(x) = x - 6$ on the interval $[-1, 3]$ is

(a) $50/3$
(b) 22
(c) 31
(d) $100/3$

A 3.4

(d) 100/3



The area of the region bounded above by $f(x) = x^2 + 1$ and below by $g(x) = x - 6$ on the interval $[-1,3]$ is equal to the sum of the areas of the regions bounded above by $f(x) = x^2 + 1$ and below by the horizontal axis on the interval $[-1,3]$ (region shaded blue), and of the one bounded above by the horizontal axis and below by $g(x) = x - 6$ on the interval $[-1,3]$ (region shaded green).
Area of the region bounded above by $f(x) = x^2 + 1$ and below by the horizontal axis on the interval $[-1,3]$ is equal to the value of the following integral

$$\int_{-1}^{3}(x^2+1)\mathrm{d}x$$
$$= \left(\frac{3^3}{3}+3\right) - \left(\frac{(-1)^3}{3}+(-1)\right)$$
$$= \frac{40}{3}$$

Area of the region bounded above by the horizontal axis and below by $g(x) = x - 6$ on the interval $[-1,3]$ is equal to the absolute value of the following integral

$$\int_{-1}^{3}(x-6)\mathrm{d}x$$
$$= \left(\frac{3^2}{2}-6(3)\right) - \left(\frac{(-1)^2}{2}-6(-1)\right)$$
$$= -20$$

Therefore, the required area = $\frac{40}{3} + 20 = \frac{100}{3}$

Exercise 3.5 $\lim_{x\to 3}\left(\frac{x^2-5x-2}{x-2}\right)^{1/3}$ equals

(a) -2
(b) $-4/3$
(c) $(-4/3)^{1/3}$
(d) $(2/3)^{1/3}$

A 3.5

(a) -2

$$\lim_{x\to 3}\left(\frac{x^2-5x-2}{x-2}\right)^{1/3} = \left(\frac{3^2-5(3)-2}{3-2}\right)^{1/3} = (-8)^{1/3} = -2$$

Exercise 3.6 Consider an individual A with utility function $u(x) = 10\sqrt{x}$, where x denotes the amount of money available to her. Suppose, she has Rs 100. However, she has option of buying a lottery that will cost her Rs 51. If purchased, the lottery pays Rs 351 with probability p, and pays 0 (nothing) with remaining probability. Assume that A is expected utility maximizer. Which of the following statements is correct? A will

(a) not prefer to buy the lottery at all as long as $p < 1$
(b) certainly prefer to buy the lottery as long as $p > 0$
(c) prefer to buy the lottery if and only if $p > 51/351$
(d) prefer to buy the lottery if and only if $p > 51/221$

A 3.6

(d) prefer to buy the lottery if and only if $p > 51/221$

She will buy the lottery if her expected utility from buying the lottery is more than her utility from not buying the lottery.

$$\begin{aligned}
& p\cdot u(100-51+351)+(1-p)\cdot u(100-51) > u(100)\\
\Leftrightarrow\quad & p\cdot u(400)+(1-p)\cdot u(49) > u(100)\\
\Leftrightarrow\quad & p\cdot 10\sqrt{400}+(1-p)\cdot 10\sqrt{49} > 10\sqrt{100}\\
\Leftrightarrow\quad & p\cdot 200+(1-p)\cdot 70 > 100\\
\Leftrightarrow\quad & p\cdot 130 > 30\\
\Leftrightarrow\quad & p > 3/13\\
\Leftrightarrow\quad & p > 51/221
\end{aligned}$$

Answer 7 and 8 for the following situation:

Suppose that there are two goods, which are imperfect substitutes of each other. Let p_1, p_2 denote the price of good 1 and good 2, respectively. Demand of good 1 and good 2 are as follows:

$$D_1(p_1,p_2) = a - p_1 + bp_2 \quad D_2(p_1,p_2) = a - p_2 + bp_1$$

where $a > 0$ and $1 > b > 0$. Both of the goods can be produced at cost c per unit.

Exercise 3.7 Find the equilibrium prices, when good 1 and good 2 are produced by two different monopolists

(a) $p_1 = p_2 = \dfrac{a+c}{2-b}$
(b) $p_1 = p_2 = \dfrac{a+c}{1-b}$

(c) $p_1 = \dfrac{a+c}{2-b}, p_2 = \dfrac{a+c}{1-b}$

(d) $p_1 = \dfrac{a+c}{1-b}, p_2 = \dfrac{a+c}{2-b}$

A 3.7

(a) $p_1 = p_2 = \dfrac{a+c}{2-b}$

To solve for the equilibrium prices, we first solve for the best response functions of the two monopolists. Monopolist producing commodity $i \in \{1,2\}$ maximises its profit taking as given the price charged by the other firm $j(\neq i) \in \{1,2\}$:

$$\max_{p_i} \quad \pi_i(p_i, p_j) = (p_i - c)(a - p_i + bp_j)$$

Differentiating π_i with respect to p_i,

$$\frac{\partial \pi_i}{\partial p_i}(p_i, p_j) = (a - p_i + bp_j) - (p_i - c)$$

The first order condition $\dfrac{\partial \pi_i}{\partial p_i}(p_i, p_j) = 0$ leads to the best response function of firm i,

$$\text{BR}_i(p_j) = \frac{a + bp_j + c}{2}$$

Solving $\text{BR}_1(p_2)$ and $\text{BR}_2(p_1)$ simultaneously, we get the following equilibrium prices

$$p_1 = p_2 = \frac{a+c}{2-b}$$

Exercise 3.8 Find the equilibrium prices, when both the goods are produced by single monopolist.

(a) $p_1 = p_2 = \dfrac{a+c-bc}{2-b}$

(b) $p_1 = p_2 = \dfrac{a+c-bc}{1-b}$

(c) $p_1 = p_2 = \dfrac{a+c-bc}{2(1-b)}$

(d) $p_1 = p_2 = \dfrac{a+c-bc}{2}$

A 3.8

(c) $p_1 = p_2 = \dfrac{a+c-bc}{2(1-b)}$

To solve for the equilibrium prices, the joint monopolist maximises its combined profits:

$$\max_{p_1,p_2} \quad \pi(p_1,p_2) = (p_1-c)(a-p_1+bp_2)+(p_2-c)(a-p_2+bp_1)$$

Differentiating π with respect to p_1 and p_2,

$$\frac{\partial \pi}{\partial p_1}(p_1,p_2) = (a-p_1+bp_2)-(p_1-c)+b(p_2-c)$$
$$\frac{\partial \pi}{\partial p_2}(p_1,p_2) = (a-p_2+bp_1)-(p_2-c)+b(p_1-c)$$

Solving the first order conditions $\dfrac{\partial \pi}{\partial p_1}(p_1,p_2)=0$ and $\dfrac{\partial \pi}{\partial p_2}(p_1,p_2)=0$ gives the following as equilibrium prices,

$$p_1 = p_2 = \frac{a+c-bc}{2(1-b)}$$

Answer 9 and 10 using the following information:

H stands for the Headcount Ratio of Poverty (total number of poor divided by total population); C for Mean Consumption per-capita; E for Elasticity of H with respect to C; NSS for National Sample Survey; and NAS for National Accounts Statistics.

Table 3.1: (Source: Datt and Ravallion; EPW 2010)

	Rate of Change of H (% per annum)	Rate of Population Growth (% per annum)
Pre-1991	-1.1	2.2
Post-1991	-2.4	1.7

Table 3.2: (Source: Datt and Ravallion; EPW 2010)

	E when C is based on NSS data	E when C is based on NAS data
Pre-1991	-1.6	-1.0
Post-1991	-2.1	-0.7

Exercise 3.9 From the data in Table 1, we can conclude that the total number of poor people was

(a) Rising before 1991 but falling after 1991
(b) Falling before 1991 but rising after 1991
(c) Falling both before and after 1991, but at different rates
(d) Rising both before and after 1991, but at different rates

A 3.9

(a) Rising before 1991 but falling after 1991

Let $n(t)$ denotes the level of population in the country at time t, then it is given that

$$n(t+1) = \begin{cases} 1.022\,n(t) & \text{for } t < 1991 \\ 1.017\,n(t) & \text{for } t \geq 1991 \end{cases}$$

Let $p(t)$ denotes the number of poor people in the country at time t, then the following information is provided about the head count ratio at time t (denoted by $\mathrm{H}(t)$),

$$\mathrm{H}(t) = \frac{p(t)}{n(t)}; \quad \mathrm{H}(t+1) = \begin{cases} 0.989\,\mathrm{H}(t) & \text{for } t < 1991 \\ 0.976\,\mathrm{H}(t) & \text{for } t \geq 1991 \end{cases}$$

We are interested in finding out that whether the fraction $\frac{p(t+1)}{p(t)}$ was greater than or less than one in the two phases. For $t < 1991$,

$$\frac{p(t+1)}{p(t)} = \frac{\mathrm{H}(t+1) \times n(t+1)}{\mathrm{H}(t) \times n(t)} = \frac{\mathrm{H}(t+1)}{\mathrm{H}(t)} \times \frac{n(t+1)}{n(t)} = 0.989 \times 1.022 = 1.01$$

For $t \geq 1991$,

$$\frac{p(t+1)}{p(t)} = \frac{\mathrm{H}(t+1) \times n(t+1)}{\mathrm{H}(t) \times n(t)} = \frac{\mathrm{H}(t+1)}{\mathrm{H}(t)} \times \frac{n(t+1)}{n(t)} = 0.976 \times 1.017 = 0.992$$

Therefore, the total number of poor people was rising before 1991 but falling after 1991.

Exercise 3.10 From the data in Table 2, we can conclude that both before and after 1991, mean consumption per-capita according to NAS was:

(a) Lower than mean consumption per-capita according to NSS
(b) Growing faster than mean consumption per-capita according to NSS
(c) Growing slower than mean consumption per-capita according to NSS
(d) None of the above

A 3.10

(b) Growing faster than mean consumption per-capita according to NSS

Elasticity of H with respect to C is $\mathrm{E} = \frac{\Delta \mathrm{H}}{\Delta \mathrm{C}} \times \frac{\mathrm{C}}{\mathrm{H}}$. We are going to compare the elasticities in the two phases according to NAS data and NSS data to draw the required conclusions about the mean consumption per-capita.

For Pre-1991,

$$\mathrm{E}_{\mathrm{NAS}} = \frac{\Delta \mathrm{H}}{\Delta \mathrm{C}} \times \frac{\mathrm{C}}{\mathrm{H}} = -1 \quad \Rightarrow \quad \frac{\Delta \mathrm{C}}{\mathrm{C}} = -\frac{\Delta \mathrm{H}}{\mathrm{H}} = 0.011$$
$$\mathrm{E}_{\mathrm{NSS}} = \frac{\Delta \mathrm{H}}{\Delta \mathrm{C}} \times \frac{\mathrm{C}}{\mathrm{H}} = -1.6 \quad \Rightarrow \quad \frac{\Delta \mathrm{C}}{\mathrm{C}} = -\frac{\Delta \mathrm{H}}{\mathrm{H}} \times \frac{1}{1.6} = 0.0068$$

For Post-1991,

$$\mathrm{E}_{\mathrm{NAS}} = \frac{\Delta \mathrm{H}}{\Delta \mathrm{C}} \times \frac{\mathrm{C}}{\mathrm{H}} = -0.7 \quad \Rightarrow \quad \frac{\Delta \mathrm{C}}{\mathrm{C}} = -\frac{\Delta \mathrm{H}}{\mathrm{H}} \times \frac{1}{0.7} = \frac{0.024}{0.7} = 0.034$$
$$\mathrm{E}_{\mathrm{NSS}} = \frac{\Delta \mathrm{H}}{\Delta \mathrm{C}} \times \frac{\mathrm{C}}{\mathrm{H}} = -2.1 \quad \Rightarrow \quad \frac{\Delta \mathrm{C}}{\mathrm{C}} = -\frac{\Delta \mathrm{H}}{\mathrm{H}} \times \frac{1}{2.1} = \frac{0.024}{2.1} = 0.0114$$

Therefore, we can conclude that both before and after 1991, mean consumption per-capita according to NAS was growing faster than mean consumption per-capita according to NSS.

Exercise 3.11 If two balanced die are rolled, the sum of dots obtained is even with probability

(a) $1/2$
(b) $1/4$
(c) $3/8$
(d) $1/3$

A 3.11

(a) $1/2$

Let X_1 be the number of dots on dice 1, X_2 be the number of dots on dice 2. Note that X_1 and X_2 are independent. We want to find the probability that the sum of the dots is even.

$$\begin{aligned}
\Pr(X_1 + X_2 \text{ is even}) &= \Pr(X_1 \text{ is even}, X_2 \text{ is even}) + \Pr(X_1 \text{ is odd}, X_2 \text{ is odd}) \\
&= [\Pr(X_1 \text{ is even}) \times \Pr(X_2 \text{ is even})] + \\
&\quad [\Pr(X_1 \text{ is odd}) \times \Pr(X_2 \text{ is odd})] \\
&= [\Pr(X_1 \in \{2,4,6\}) \times \Pr(X_2 \in \{2,4,6\})] + \\
&\quad [\Pr(X_1 \in \{1,3,5\}) \times \Pr(X_2 \in \{1,3,5\})] \\
&= \left[\frac{1}{2} \times \frac{1}{2}\right] + \left[\frac{1}{2} \times \frac{1}{2}\right] \\
&= \frac{1}{2}
\end{aligned}$$

Exercise 3.12 A population is growing at the instantaneous growth rate of 1.5 per cent. The time taken (in years) for it to double is approximately

(a) $\frac{\log 2}{0.15}$

(b) $\dfrac{\log 2}{15}$
(c) $\dfrac{\log 2}{0.015}$
(d) $\dfrac{\log 2}{1.5}$

A 3.12

(c) $\dfrac{\log 2}{0.015}$

The time T taken for the population to double if it is growing at the instantaneous growth rate of 1.5 per cent solves

$$e^{0.015T} = 2 \quad \Leftrightarrow \quad T = \frac{\log 2}{0.015}$$

Exercise 3.13 A linear regression model $y = \alpha + \beta x + \varepsilon$ is estimated using OLS. It turns out that the estimated $\hat{\beta}$ equals zero. This implies that:
(a) R^2 is zero
(b) R^2 is one
(c) $0 < R^2 < 1$
(d) In this case R^2 is undefined

A 3.13

(a) R^2 is zero

$$\begin{aligned}
R^2 &= \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2 \sum_{i=1}^{n}(x_i - \bar{x})^2}}\right]^2 \\
&= \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]^2 \times \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\right] \\
&= \left[\hat{\beta}\right]^2 \times \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\right] \\
&= 0
\end{aligned}$$

Exercise 3.14 An analyst has data on wages for 100 individuals. The arithmetic mean of the log of wages is the same as:
(a) Log of the geometric mean of wages
(b) Log of the arithmetic mean of wages
(c) Exponential of the arithmetic mean of wages
(d) Exponential of the log of arithmetic mean of wages

A 3.14

(a) Log of the geometric mean of wages

Let $w_1, w_2, \ldots, w_{100}$ be the data of wages for 100 individuals.

$$\begin{aligned}
\text{A.M.}(\log w) &= \frac{\sum_{i=1}^{100} \log w_i}{100} \\
&= \sum_{i=1}^{100} \frac{\log w_i}{100} \\
&= \sum_{i=1}^{100} \log\left(w_i^{\frac{1}{100}}\right) \\
&= \log\left(\prod_{i=1}^{100}\left(w_i^{\frac{1}{100}}\right)\right) \\
&= \log\left(\left(\prod_{i=1}^{100} w_i\right)^{\frac{1}{100}}\right) \\
&= \log\left(\text{G.M.}(w)\right)
\end{aligned}$$

Exercise 3.15 A certain club consists of 5 men and 5 women. A 5 member committee consisting of 2 men and 3 women has to be constituted. How many ways are there of constituting this committee?

(a) 20
(b) 100
(c) 150
(d) None of the above

A 3.15

(b) 100

$$\begin{aligned}
& \text{Number of ways of constituting this committee} \\
= {} & \text{Number of ways of choosing 2 men from 5 men} \times \\
& \text{Number of ways of choosing 3 women from 5 women} \\
= {} & \binom{5}{2} \times \binom{5}{3} \\
= {} & 100
\end{aligned}$$

Exercise 3.16 Exchange rate overshooting occurs:

(a) under fixed exchange rates when the central bank mistakenly buys or sells too much foreign exchange
(b) under fixed exchange rates as a necessary part of the adjustment process for any monetary shock
(c) under flexible exchange rates when the exchange rate rises (depreciates) above and then falls down to equilibrium after a monetary expansion
(d) decreases the value of the determinant

A 3.16

(c) under flexible exchange rates when the exchange rate rises (depreciates) above and then falls down to equilibrium after a monetary expansion

An increase in the money supply (or decline in the demand for money) shifts LM to the right to LM' causing domestic residents to re-balance their portfolios by purchasing assets abroad. The domestic currency devalues, reducing the real exchange rate, shifting world demand onto domestic goods and increasing the level of output the short run when the price level is inflexible. The IS curve shifts rightward to IS' . The domestic unemployment rate falls below its normal (full-employment) level. This pressure on the labour market eventually causes nominal wages and prices to rise.

As the price level rises the real money stock declines, shifting LM back to its original position. This rise in the price level reduces the real exchange rate back to its original level, shifting IS back to its original position. When full employment is again achieved the price level and nominal exchange rate will have risen in proportion to the increase in the money supply and the levels of real output and the real exchange rate will have returned to their original levels. The monetary expansion thus has a temporary downward effect on the real exchange rate and upward effect on real income and a permanent upward effect on the nominal exchange rate and domestic price level.

The above analysis assumes that the short-run adjustment of output and employment is immediate while the adjustment in the price level takes time. In fact, however, the adjustment of output will also take time—it will take time for the devaluation of the real and nominal exchange rates to shift world demand onto domestic goods and a further period for the increased demand to increase domestic output. This has important implications for the process of exchange rate adjustment.

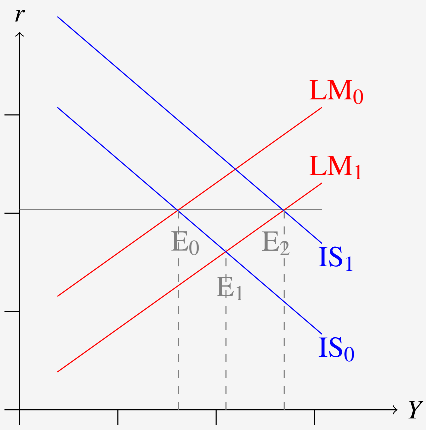
Exercise 3.17 In an open economy with a system of flexible exchange rates and perfect capital mobility, an expansionary monetary policy:

(a) causes the domestic currency to appreciate
(b) has a greater impact on income than in a closed economy
(c) increases capital inflows into the country
(d) induces a balance of payments deficit

A 3.17

(b) has a greater impact on income than in a closed economy

Expansionary monetary policy will shift LM from LM_0 to the right LM_1- there is an increase in Y and a fall in r. As a result there is an outflow of capital (domestic interest rate is lower than international one) causing depreciation of domestic currency. Depreciation leads to an increase in exports that shifts IS from IS_0 to IS_1. Final result is higher Y, constant r. Therefore, monetary policy is effective.



In the above diagram E_0 is the initial equilibrium, E_1 is the new equilibrium in the closed economy and E_2 is the new equilibrium in the open economy.

Exercise 3.18 The short-run aggregate supply curve is upward sloping because

(a) lower price level creates a wealth effect
(b) lower taxes motivate people to work more
(c) money wages do not immediately change when the price level changes
(d) most business firms operate with long-term contracts for output but not labour

A 3.18

(c) money wages do not immediately change when the price level changes

Let the production function be $Y = F(N, \overline{K})$, where $\overline{K}$ is fixed. Let $N^d(W/P)$ be the downward sloping labor demand function and $N^s(W/P)$ be the upward sloping labor supply supply function. Now, wages are usually rigid downwards in the short run (may not be upwards). If money wages do not respond to fall in price (in the short run), then the employment magnitude of the new employment will be determined by the labor demand as $N^d(W/P) < N^s(W/P)$. Since price has fallen, real wage rises and therefore, demand for labor and hence the equilibrium employment falls, further causing the output to fall. So, output falls as a result of fall in price and therefore we get upward sloping short-run aggregate supply.

Exercise 3.19 The term 'seignorage' is associated with

(a) inflation generated by printing new money
(b) real revenue created by printing new money
(c) public indebtedness created by printing new money

(d) none of the above

A 3.19

(b) real revenue created by printing new money

Seignorage is the difference between the value of money and its cost of production. It is the “gain" of the central bank from printing more money. When the central bank lends to the commercial banks, it gains from the interest income. Also, inflation leads to rise in the value of assets that the bank holds such as gold, silver, etc., relative to its liabilities (high powered money) and this is also part of Seignorage.

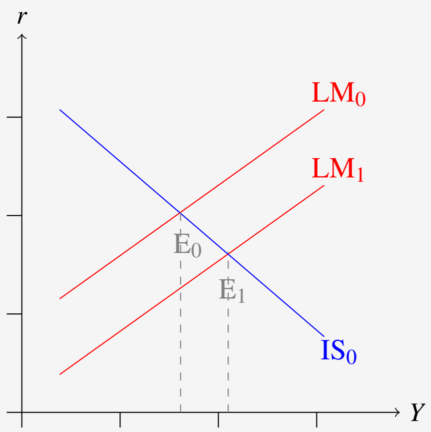
Exercise 3.20 If money demand is stable, an open market purchase of government securities by the central bank will:

(a) increase both the level of income and the interest rate
(b) decrease both the level of income and the interest rate
(c) increase the level of income and lower the interest rate
(d) decrease the level of income and raise the interest rate

A 3.20

(c) increase the level on income and lower the interest rate

If money demand is stable, an open market purchase of government securities by the central bank will lead to rise in Money supply and cause the LM curve to shift to the right. This will lead to fall in interest rate and rise in income.



In the above diagram E_0 is the initial equilibrium and E_1 is the new equilibrium resulting from rise in money supply.

Exercise 3.21Q 21 The function defined by $f(x) = x^5 + 7x^3 + 13x - 18$

(a) may have 5 real roots
(b) has no real roots
(c) has 3 real roots
(d) has exactly 1 real root

A 3.21

(d) has exactly 1 real root

By differentiating $f(x) = x^5 + 7x^3 + 13x - 18$, we get

$$f'(x) = 5x^4 + 21x^2 + 13 > 0 \quad \forall x \in \mathbb{R}$$

Therefore, f is strictly increasing and continuous further implying that f has at most one root. Also, $f(0) = -18$ and $f(1) = 3$, this further implies that there is one root between 0 and 1 (by Intermediate Value Theorem).

Exercise 3.22 The function $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^3 - 3x^2 + 6$ is

(a) concave on $(-\infty, 2)$ and convex on $(2, \infty)$.
(b) concave on $(-1, 2)$, convex on $(-\infty, -1)$ and $(2, \infty)$.
(c) convex on $(-1, 2)$, concave on $(-\infty, -1)$ and $(2, \infty)$.
(d) convex on $(-\infty, 2)$ and concave on $(2, \infty)$.

A 3.22

(b) concave on $(-1, 2)$, convex on $(-\infty, -1)$ and $(2, \infty)$.

Differentiating $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^3 - 3x^2 + 6$ twice, we get

$$f''(x) = 3x^2 - 3x - 6 = 3(x-2)(x+1) \begin{cases} > 0 & \text{for } x \in (-\infty, -1) \\ < 0 & \text{for } x \in (-1, 2) \\ > 0 & \text{for } x \in (2, \infty) \end{cases}$$

Therefore, f is concave on $(-1, 2)$, convex on $(-\infty, -1)$ and $(2, \infty)$.

Exercise 3.23 Consider the function $f(x) = \frac{9}{2}x^{(2/3)} - \frac{3}{5}x^{(5/3)}$ for all x in the closed interval $[-1, 5]$. $f(5)$ is approximately equal to 4.386. $f(x)$ attains a maximum on interval $[-1, 5]$ at

(a) $x = -1$
(b) $x = 2$
(c) $x = 3$
(d) $x = 4$

A 3.23

(c) $x = 3$

Note that $f(x) = \frac{9}{2}x^{(2/3)} - \frac{3}{5}x^{(5/3)}$ is a concave function because it can be written as a sum of two concave functions $g(x) = \frac{9}{2}x^{(2/3)}$ and $h(x) = -\frac{3}{5}x^{(5/3)}$. This implies that if there exist $x^* \in [-1, 5]$ such that $f'(x^*) = 0$, then f attains a maximum at x^*. Differentiating f, we get

$$f'(x) = 3x^{(-1/3)} - x^{(2/3)}$$

Clearly, $f'(3) = 0$.

Exercise 3.24 Consider the function f defined by $f(x) = x^6 + 5x^4 + 2$, for all $x \geq 0$. The derivative of its inverse function evaluated at $f(x) = 8$, that is, $f(^{-1})'(8)$ equals

(a) $1/7$
(b) $1/15$
(c) $1/26$
(d) $1/20$

A 3.24

(c) $1/26$

Let us first derive the expression for the derivative of the inverse function. Define $g(x) \equiv f^{-1}(x)$ or equivalently, $x = f(g(x))$. Differentiating with respect to x, we get

$$
\begin{aligned}
& 1 = f'(g(x))g'(x) \\
\Leftrightarrow \quad & g'(x) = \frac{1}{f'(g(x))} \\
\Leftrightarrow \quad & f(^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\end{aligned}
$$

Therefore,

$$
f(^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(1)} = \frac{1}{26}
$$

Exercise 3.25 Consider the following functions. $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x,y) = (x+2y, x-y, -2x+3y)$. And $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x,y) = (x+1, y+2)$. Then

(a) Both f and g are linear transformations
(b) f is a linear transformation, but g is not a linear transformation
(c) f is not a linear transformation, but g is a linear transformation
(d) Neither f nor g is a linear transformation

A 3.25

(b) f is a linear transformation, but g is not a linear transformation.

$h : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a *linear transformation* if for any two vectors v and v' in $\mathbb{R}^n$ and any scalar α in $\mathbb{R}$, the following two conditions hold,

$$\begin{aligned} h(v+v') &= h(v)+h(v') \\ h(\alpha v) &= \alpha h(v) \end{aligned}$$

We will now show that $f(x,y) = (x+2y, x-y, -2x+3y)$ is a linear transformation but $g(x,y) = (x+1, y+2)$ is not.

$$\begin{aligned} f(x,y)+f(x',y') &= (x+2y, x-y, -2x+3y)+(x'+2y', x'-y', -2x'+3y') \\ &= ((x+x')+2(y+y'), (x+x')-(y+y'), -2(x+x')+3(y+y')) \\ &= f(x+x', y+y') \\ &= f((x,y)+(x',y')) \\ f(\alpha(x,y)) &= f(\alpha x, \alpha y) \\ &= (\alpha x+2\alpha y, \alpha x-\alpha y, -2\alpha x+3\alpha y) \\ &= \alpha(x+2y, x-y, -2x+3y) \\ &= \alpha f(x,y) \end{aligned}$$

Therefore, f is a linear transformation.

For $\alpha \neq 1$,

$$\begin{aligned} g(\alpha(x,y)) &= g(\alpha x, \alpha y) \\ &= (\alpha x+1, \alpha y+2) \\ &\neq (\alpha x+\alpha, \alpha y+2\alpha) \\ &= \alpha(x+1, y+2) \\ &= \alpha g(x,y) \end{aligned}$$

Therefore, g is not a linear transformation.

Exercise 3.26 A six meter long string is cut in two pieces. The first piece, with length equal to some x, is used to make a circle, the second, with length $(6-x)$, to make a square. What value of x will minimize the sum of the areas of the circle and the square? (x is allowed to be 0 or 6 as well).

(a) $x = 24\pi/(1+4\pi)$
(b) $x = 6\pi/(4+\pi)$
(c) $x = 6$
(d) $x = 1/2\pi$

A 3.26

(b) $x = 6\pi/(4+\pi)$

The piece with length equal to x will make a circle of radius $x/(2\pi)$ and the piece with length $(6-x)$ will make a square of side $(6-x)/4$. Therefore, we minimize the sum of the areas with respect to x,

$$\min_{0\le x\le 6} \quad \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{6-x}{4}\right)^2$$

Differentiating the objective with respect to x, we get the first order condition as

$$\frac{x}{2\pi} - \frac{1}{8}(6-x) = 0$$

Solving for x, we get

$$x = \frac{6\pi}{4+\pi}$$

■

Exercise 3.27 The repeated nonterminating decimal $0.272727\ldots$

(a) cannot be represented as a fraction.
(b) equals $27/99$
(c) lies strictly between $27/99$ and $27/100$.
(d) is an irrational number.

A 3.27

(b) equals $27/99$

Let $x = 0.272727\ldots$. Then, $100x = 27.272727\ldots$. Therefore, $100x - x = 27$. This gives us $x = 27/99$.

■

Answer 28, 29 and 30 based on the following situation:
Suppose three players, 1, 2 and 3, use the following procedure to allocate 9 indivisible coins. Player 1 proposes an allocation (x_1, x_2, x_3) where x_i is the number of coins given to player i. Players 2 and 3 vote on the proposal, saying either Y (Yes) or N (No). If there are two Y votes, then the proposed allocation is implemented. If there are two N votes, the proposal is rejected. If there is one Y vote and one N vote, then player 1 gets to vote Y or N. Now, the proposal is accepted if there are two Y votes and rejected if there are two N votes.

If 1's proposal is rejected, then 2 makes a proposal. Now, only 3 votes Y or N. If 3 votes Y, then 2's proposal is accepted. If 3 votes N, then the proposal is rejected and the allocation (3, 3, 3) is implemented.

Assume that, if the expected allocation to be received by a particular player by voting Y or N is identical, then the player votes N.

Exercise 3.28 If 1's proposal is rejected and 2 gets to make a proposal, her proposal will be

(a) $(0,5,4)$
(b) $(0,4,5)$
(c) $(0,6,3)$

(d) $(0,3,6)$

A 3.28

(a) $(0,5,4)$

If 1's proposal is rejected and 2 gets to make a proposal, her proposal will be $(0,5,4)$. This is because 3 will reject any offer in which she will get 3 coins or less, and if 2's proposal is rejected by 3, then the allocation $(3,3,3)$ is implemented. So, 2's offer that maximise his own payoff is the one that offers nothing to individual 1 and offers the smallest number of coins that remains acceptable to individual 3 i.e an offer of 4 coins to individual 3 and the remainder for herself.

Exercise 3.29 1's proposal will be

(a) $(5,0,4)$
(b) $(4,0,5)$
(c) $(3,6,0)$
(d) $(6,3,0)$

A 3.29

(b) $(4,0,5)$

From Q 28 we know that if 1's offer is rejected in the first stage then $(0,5,4)$ will be the resulting payoffs. So, 1 will never want its offer to get rejected in the first stage. Now 1's proposal in the first stage will get accepted if at least one of the two remaining players accepts her offer. In order to make the offer "attractive" for player 2, player 1 must offer at least 6 coins to her and make the offer attractive for player 3, she must offer at least 5 coins to her. Therefore, the proposal that maximises player 1's payoff will be the one in which she offers 5 coins to player 3 and nothing to player 2 so that even if player 2 rejects her offer, player 3 accepts it and then player 1 will accept her own offer so that $(4,0,5)$ is implemented.

Exercise 3.30 Consider the following change of the above situation. If 2 makes a proposal and 3 votes Y, then 2's proposal is implemented. However, if 3 votes N, then 1 gets to choose between 2's proposal and the allocation $(3,3,3)$. If 1's proposal is rejected and 2 gets to make a proposal, her proposal will be

(a) $(4,5,0)$
(b) $(0,5,4)$
(c) either (a) or (b)
(d) neither (a) nor (b)

A 3.30

(c) either (a) or (b)

If 1's proposal is rejected and 2 gets to make a proposal, her proposal can be $(0,5,4)$ or $(4,5,0)$. This is because for 2's proposal to be implemented she must "please" either player 3 or player 1 as both can get 3 each if they choose to reject 2's proposal. So, they will accept only offers of 4 or more. Since it is sufficient to please one of them, 2's proposal will be either $(0,5,4)$ or $(4,5,0)$.

Exercise 3.31 Utility of a consumer is given by $u(x_1,x_2) = \min\{x_1,x_2\}$. His income is M, and price of good 2 is 1. There are two available price schemes for good 1: (i) per unit price 2 and (ii) a reduced per unit price $2-\theta$ along with a fixed fee T. A consumer would be indifferent between the above schemes if

(a) $\theta = 2T/M$
(b) $\theta = 3T/M$
(c) $\theta = T/M$
(d) $\theta = (T+1)/M$

A 3.31

(b) $\theta = 3T/M$

If the price scheme is (i), consumer's optimal choice (x_1,x_2) satisfy $x_1 = x_2$ and the budget constraint $2x_1 + x_2 = M$, therefore the optimal choice is $(x_1,x_2) = \left(\frac{M}{3},\frac{M}{3}\right)$ and the utility at this consumption level is $\frac{M}{3}$.

If the price scheme is (ii), consumer's optimal choice (x_1,x_2) satisfy $x_1 = x_2$ and the budget constraint $(2-\theta)x_1 + x_2 = M-T$, therefore the optimal choice is $(x_1,x_2) = \left(\frac{M-T}{3-\theta},\frac{M-T}{3-\theta}\right)$ and the utility at this consumption level is $\frac{M-T}{3-\theta}$.

A consumer would be indifferent between the above schemes if $\frac{M}{3} = \frac{M-T}{3-\theta}$ or equivalently, $\theta = \frac{3T}{M}$.

Answer 32, 33 based on the following model:
Suppose, an individual lives for two periods. In each period she consumes only one good, which is rice. In period 2, she can costlessly produce 1 unit of rice, but in period 1 she produces nothing. However, in period 1 she can borrow rice at an interest rate $r > 0$. That is, if she borrows z units of rice in period 1, then in Period 2, she must return $z(1+r)$ units of rice. Let x_1 and x_2 denote her consumption of rice in period 1 and period 2, respectively; $x_1, x_2 \geq 0$. Her utility function is given by $U(x_1,x_2) = x_1 + \beta x_2$, where β is the discount factor, $0 < \beta < 1$. Note that there are only two sources through which rice can be available; own production and borrowing.

Exercise 3.32 Find the interest rate r, at which the individual would borrow $\frac{1}{2}$ unit of rice in period 1.

(a) 2
(b) $\frac{1}{2}$
(c) $\beta - 1$
(d) $\frac{1}{\beta} - 1$

A 3.32

(d) $\frac{1}{\beta} - 1$

Utility maximization problem of the consumer in question is

$$\max_{(x_1,x_2)\geq 0} \quad x_1 + \beta x_2$$
$$\text{s.t.} \quad x_1(1+r) + x_2 = 1$$

For $x_1 > 0$ in optimum, it must be the case that $r \leq \frac{1}{\beta} - 1$. Putting it more precisely, he will demand $x_1 = \frac{1}{1+r}$ if $r < \frac{1}{\beta} - 1$. And he can pick any amount of x_1 from the closed interval $[0, \frac{1}{1+r}]$ when $r = \frac{1}{\beta} - 1$. Rewriting this interval in terms of β, we get that he can pick any amount of x_1 from the closed interval $[0, \beta]$. Now it is given that the individual consumes $\frac{1}{2}$ unit of rice in period 1, so there are two possibilities

1. If $r < \frac{1}{\beta} - 1$, then $x_1 = \frac{1}{1+r}$, and we also know $x_1 = \frac{1}{2}$. Therefore, he will consume $x_1 = \frac{1}{2}$ when $r = 1$ and $\beta < \frac{1}{2}$.
2. If $r = \frac{1}{\beta} - 1$, then he can consume $x_1 = \frac{1}{2}$ when $r = \frac{1}{\beta} - 1$ and $\beta \geq \frac{1}{2}$.

So, the answer is either $r = 1$ or $r = \frac{1}{\beta} - 1$.

Exercise 3.33 Now suppose that there are N agents in the above two period economy. The agents are identical (in terms of production and utility function) except that they have different discount factors. Suppose that β follows *uniform distribution* in the interval $\left[\frac{1}{2}, 1\right]$. Assuming $r \leq 1$, the demand function for rice in period 1 will be

(a) $N\frac{(1-r)}{(1+r)}$
(b) $N\frac{(1-r)}{(1+r)^2}$
(c) $N\frac{r}{1+r}$
(d) $N\frac{r}{2}$

A 3.33

(b) $N\frac{(1-r)}{(1+r)^2}$

If $r < \frac{1}{\beta} - 1$ or equivalently, $\beta < \frac{1}{1+r}$, then $x_1 = \frac{1}{1+r}$. So, expected demand is

$$N\int_{\frac{1}{2}}^{\frac{1}{1+r}} \frac{2}{1+r} d\beta = \frac{2N}{1+r}\left[\frac{1}{1+r} - \frac{1}{2}\right] = \frac{N(1-r)}{(1+r)^2}$$

Answer 34, 35 based on the following situation:
Consider a two-person two-good exchange economy: persons/agents are A and B, and goods are 1 and 2. The agents have the following utility functions:

$$u_A(x_1, x_2) = \alpha x_1 + x_2 \quad u_B(y_1, y_2) = y_1 y_2$$

where x_1 and x_2 denote the allocation to A of good 1 and good 2, respectively. Similarly, y_1 and y_2 denote the allocation to B of good 1 and good 2, respectively. There are 5 units of each good; i.e., $x_1 + y_1 = 5$ and $x_2 + y_2 = 5$.

Now, consider the following allocation: Agent A gets 4 units of good 1 only, but agent B

gets 1 unit of good 1 and 5 units of good 2.

Exercise 3.34 Suppose an agent i is said to envy agent j, if i strictly prefers j's allocation over her own allocation. And, an allocation is called 'No-envy allocation' if none of the agents envies the other. In that case,

(a) the above allocation is always 'No-envy allocation'
(b) the above allocation is never 'No-envy allocation'
(c) the above allocation is 'No-envy allocation' if $\alpha \geq \frac{5}{3}$
(d) the above allocation is 'No-envy allocation' if $\alpha \leq \frac{3}{5}$

A 3.34

(c) the above allocation is 'No-envy allocation' if $\alpha \geq \frac{5}{3}$

At $(x_1, x_2) = (4, 0)$ and $(y_1, y_2) = (1, 5)$, $u_A = 4\alpha$ and $u_B = 5$.

Clearly, B does not envy A at above allocation because $u_B(4,0) = 0 < u_B(1,5) = 5$

And A will not envy B at above allocation if $u_A(4,0) \geq u_A(1,5)$

$\implies 4\alpha \geq \alpha + 5 \implies \alpha \geq \frac{5}{3}$

Therefore, the above allocation is 'No-envy allocation' if $\alpha \geq \frac{5}{3}$

Exercise 3.35 The above allocation is

(a) always Pareto optimal
(b) never Pareto optimal
(c) Pareto optimal if $\alpha \leq 5$
(d) Pareto optimal if $\alpha \geq 5$

A 3.35

(d) Pareto optimal if $\alpha \geq 5$

The above allocation is Pareto optimal if $MRS_A \geq MRS_B$ at $((4,0),(1,5))$ i.e. $\alpha \geq 5$.

Answer 36, 37 based on the following situation:

There are two goods: a basic good, say a car, and a complementary good, say car audio. Suppose that the basic good is produced by a monopolist at no cost and the complementary good is produced by a competitive industry at cost c per unit. Let p be the price of the basic good. Each consumer has three choices:

(i) consume nothing, which gives 0 utility
(ii) consume one unit of the basic good, which gives $v - p$ utility
(iii) consume one unit of the basic good and one unit of the complementary good (**called bundle**), which gives $w - p - c$ utility.

Assume $w > v > 0$.

Next, suppose there are two types of consumers of cars:

Middle Class : They have valuations v_1 and w_1 for the basic good and the bundle, respectively.
Rich : They have valuations v_2 and w_2 for the basic good and the bundle, respectively.

Suppose, $v_2 > v_1$ and $w_2 - v_2 > c > w_1 - v_1$.

Exercise 3.36 Find the socially efficient consumption.
(a) Rich choose (ii) and Middle class choose (i)
(b) Rich choose (iii) and Middle class choose (i)
(c) Rich choose (iii) and Middle class choose (ii)
(d) Both Rich and Middle class choose (ii)

A 3.36

(c) Rich choose (iii) and Middle class choose (ii)

Since $v_2 > v_1 > 0$ and marginal cost of car is 0, and $w_2 - v_2 > c > w_1 - v_1$ and marginal cost of car audio is c, therefore in a surplus maximizing consumption the Rich must choose the bundle and Middle class chooses the car.

Exercise 3.37 Suppose that the monopolist can distinguish between two types of consumers. What prices would she charge?
(a) v_2 from Rich, and v_1 from Middle Class
(b) $w_2 - c$ from Rich, and $w_1 - c$ from Middle Class
(c) $w_1 - c$ from Rich as well as Middle Class
(d) $w_2 - c$ from Rich, and v_1 from Middle Class

A 3.37

(d) $w_2 - c$ from Rich, and v_1 from Middle Class

Monopolist chooses the price vector (p_1, p_2) to charge from the two types of consumers so that he maximizes his profit. He can do so by charging $p_i = \max\{w_i - c, v_i\}$ from type $i \in \{1,2\}$. Therefore, the monopolist will charge $w_2 - c$ from Rich, and v_1 from Middle Class.

Exercise 3.38 Suppose that a city can be described by an interval [0, 1]. Only two citizens, A and B, live in this city at different locations; A at 0.2 and B at 0.7. Government has decided to set up a nuclear power plant in this city but is yet to choose its location. Each citizen wants the plant as far as possible from her home and hence both of them have the same utility function, $u(d) = d$, where d denotes the distance between the plant and home. Find the set of Pareto optimal locations for the plant.
(a) All locations in the interval [0, 1] are Pareto optimal
(b) All locations in the interval [0.2, 0.7] are Pareto optimal
(c) 0.5 is the only Pareto optimal location
(d) 0 and 1 are the only Pareto optimal locations

A 3.38

(d) 0 and 1 are the only Pareto optimal locations

Location 0 is Pareto optimal because it is farthest from B's location in the city. So, B is happiest if the location of power plant is 0. Choice of any other location for the plant is nearer to B's location than 0 and hence will make him strictly worse off.

Location 1 is Pareto optimal because it is farthest from A's location in the city. So, A is happiest if the location of power plant is 1. Choice of any other location for the plant is nearer to A's location than 1 and hence will make him strictly worse off.

Any location $x \in (0, 0.4]$ is Pareto dominated by location 0, because moving power plant from x to 0 will make B better off without making A worse off.

Any location $x \in (0.4, 1)$ is Pareto dominated by location 0, because moving power plant from x to 1 will make both of them better off.

Therefore, 0 and 1 are the only Pareto optimal locations.

Answer 39, 40 based on the following situation:
Consider an exchange economy with agents 1 and 2 and goods x and y. Agent 1 lexicographically prefers the good x: when offered two non-identical bundles of x and y, she strictly prefers the bundle with more of good x, but if the bundles have the same amount of good x, then she strictly prefers the bundle with more of good y. However, Agent 2 lexicographically prefers good y.

Exercise 3.39 Suppose 1's endowment is $(10, 0)$ and 2's endowment is $(0, 10)$. The vector (p_x, p_y) is a competitive equilibrium vector of prices *if and only if*

(a) $p_x = 1 = p_y$
(b) $p_x > 0$ and $p_y > 0$
(c) $p_x > p_y > 0$
(d) $p_y > p_x > 0$

A 3.39

(b) $p_x > 0$ and $p_y > 0$

Agent 1 lexicographically prefers the good x. Therefore, his demand function is

$$(x_1, y_1)(p_x, p_y, m_1) = \left(\frac{m_1}{p_x}, 0\right)$$

where m_1 denotes the income of 1.

Agent 2 lexicographically prefers the good y. Therefore, his demand function is

$$(x_2, y_2)(p_x, p_y, m_2) = \left(0, \frac{m_2}{p_y}\right)$$

where m_2 denotes the income of 2.

Replacing incomes by value of the endowments, $m_1 = 10p_x$ and $m_2 = 10p_y$, we get

$$\begin{aligned}(x_1, y_1)(p_x, p_y, 10p_x) &= (10, 0)\\(x_2, y_2)(p_x, p_y, 10p_y) &= (0, 10)\end{aligned}$$

Therefore, demand equals supply at all (p_x, p_y) such that $p_x > 0$ and $p_y > 0$.

Exercise 3.40 Suppose we make only one change in the above situation: Person 1 lexicographically prefers y and 2 lexicographically prefers x. The vector (p_x, p_y) is a competitive equilibrium vector of prices *if and only if*

(a) $p_x = 1 = p_y$
(b) $p_x > 0$ and $p_y > 0$
(c) $p_x > p_y > 0$
(d) $p_y = p_x > 0$

A 3.40

(d) $p_y = p_x > 0$

Agent 1 lexicographically prefers the good y. Therefore, his demand function is

$$(x_1, y_1)(p_x, p_y, m_1) = \left(0, \frac{m_1}{p_y}\right)$$

where m_1 denotes the income of 1.

Agent 2 lexicographically prefers the good x. Therefore, his demand function is

$$(x_2, y_2)(p_x, p_y, m_2) = \left(\frac{m_2}{p_x}, 0\right)$$

where m_2 denotes the income of 2.

Replacing incomes by value of the endowments, $m_1 = 10p_x$ and $m_2 = 10p_y$, we get

$$\begin{aligned}(x_1, y_1)(p_x, p_y, 10p_x) &= \left(0, \frac{10p_x}{p_y}\right)\\ (x_2, y_2)(p_x, p_y, 10p_y) &= \left(\frac{10p_y}{p_x}, 0\right)\end{aligned}$$

Therefore, demand equals supply at all (p_x, p_y) such that $p_x = p_y > 0$.

Exercise 3.41 Consider two disjoint events A and B in a sample space S. Which of the following is correct?

(a) A and B are always independent
(b) A and B cannot be independent
(c) A and B are independent if exactly one of them has positive probability
(d) A and B are independent if both of them have positive probability

A 3.41

(c) A and B are independent if exactly one of them has positive probability

Since A and B are disjoint, $\Pr(A \cap B) = 0$. For independence of A and B, it must be the case that $\Pr(A \cap B) = \Pr(A)\Pr(B)$. Therefore, at least one of A and B must have zero probability. In other words, if at most one of the two events A and B have positive probability then A and B are independent and the weaker implication follows that A and B are independent if exactly one of them has positive probability.

Exercise 3.42 A bowl contains 5 chips, 3 marked \$1 and 2 marked \$4. A player draws 2 chips at random and is paid the sum of the values of the chips. The player's expected gain (in \$) is

(a) less than 2
(b) 3
(c) above 3 and less than 4
(d) above 4 and less than 5

A 3.42

(d) above 4 and less than 5

Let X_1 be the value of the chip drawn first and X_2 be the value of the chip drawn second. Let S be the sum of the values of the chips. Therefore, $S = X_1 + X_2$.

$$\begin{aligned}
\mathbb{E}(S) &= \mathbb{E}(X_1 + X_2) \\
&= \mathbb{E}(X_1) + \mathbb{E}(X_2) \quad \text{(by linearity of expectation)} \\
&= 2\mathbb{E}(X_1) \qquad\qquad \text{(by symmetry)} \\
&= 2\left[\frac{3}{5}(1) + \frac{2}{5}(4)\right] \\
&= \frac{22}{5} \\
&= 4.4
\end{aligned}$$

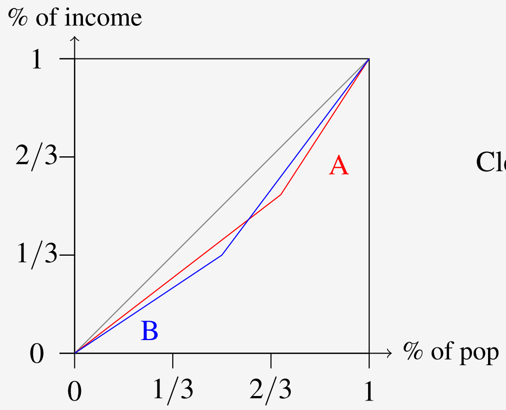
Exercise 3.43 Consider the following two income distributions in a 10 person society. A : (1000, 1000, 1000, 1000, 1000, 1000, 1000, 2000, 2000, 2000) and B : (1000, 1000, 1000, 1000, 1000, 2000, 2000, 2000, 2000, 2000). Which of the following statements most accurately describes the relationship between the two distributions?

(a) The Lorenz curve for distribution A lies to the right of that for distribution B
(b) The Lorenz curve for distribution B lies to the right of that for distribution A
(c) The Lorenz curves for the two distributions cross each other
(d) The Lorenz curves for the two distributions are identical for the bottom half of the population

A 3.43

(c) The Lorenz curves for the two distributions cross each other

One way of viewing income inequality is in terms of the Lorenz curve due-not surprisingly-to Lorenz (1905), whereby the percentages of the population arranged from the poorest to the richest are represented on the horizontal axis and the percentages of income enjoyed by the bottom $x\%$ of the population is shown on the vertical axis. Obviously 0% of the population enjoys 0% of the income and 100% of the population enjoy all the income. So a Lorenz curve runs from one corner of the unit square to the diametrically opposite corner. If everyone has the same income the Lorenz curve will be simply the diagonal, but in the absence of perfect equality the bottom income groups will enjoy a proportionately lower share of income. It is obvious, therefore, that any Lorenz curve must lie below the diagonal (except the one of complete equality which would be the diagonal), and its slope will increasingly rise-at any rate not fall-as we move to richer and richer sections of the population. In this question, we have two societies with 10 people each and data of their income distribution is as follows:
$A : (1000, 1000, 1000, 1000, 1000, 1000, 1000, 2000, 2000, 2000)$ and
$B : (1000, 1000, 1000, 1000, 1000, 2000, 2000, 2000, 2000, 2000)$.
Lorenz curve of economy A connects $(0,0) - -(0.7, 7/13) - -(1,1)$. This is because people with lower incomes i.e. 7 out of 10 in A make 7000 out of the total GDP of 13000. Lorenz curve of economy B connects $(0,0) - -(0.5, 1/3) - -(1,1)$. This is because low income group is half the population in B and makes 5000 out of the total GDP of 15000.



Clearly, the two Lorenz curves cross each other.

Exercise 3.44 A certain club consists of 5 men and 5 women. A 5-member committee consisting of 2 men and 3 women has to be constituted. Also, suppose that Mrs. F refuses to work with Mr. M. How many ways are there of constituting a 5-member committee that ensures that both of them do not work together?

(a) 50
(b) 76
(c) 108
(d) None of the above

A 3.44

(b) 76

Number of ways of choosing the 5-member committee consisting of 2 men and 3 women from a club consisting of 5 men and 5 women such that Mrs. F and Mr. M do not work together *is equal to* the number of ways of choosing the 5-member committee consisting of 2 men and 3 women from a set of 5 men and 5 women *minus* the number of ways of choosing the 5-member committee consisting of 2 men and 3 women that includes both Mrs. F and Mr. M.

$$\begin{aligned}
& \text{Number of ways of constituting such a committee} \\
= & \left[\binom{5}{2} \times \binom{5}{3}\right] - \left[\binom{4}{1} \times \binom{4}{2}\right] \\
= & \; 100 - 24 \\
= & \; 76
\end{aligned}$$

■

Exercise 3.45 Suppose, you are an editor of a magazine. Everyday you get two letters from your correspondents. Each letter is as likely to be from a male as from a female correspondent. The letters are delivered by a postman, who brings one letter at a time. Moreover, he has a 'ladies first' policy; he delivers letter from a female first, if there is such a letter. Suppose you have already received the first letter for today and it is from a female correspondent. What is the probability that the second letter will also be from a female?

(a) 1/2
(b) 1/4
(c) 1/3
(d) 2/3

A 3.45

(c) $1/3$

Let the two letters be Red (R) and Blue (B). There are four possible outcomes. Both the letters are from males, denoted MM. Letter R from a male and B from a female, denoted MF. Letter R from a female and B from a male, denoted FM. And finally, both the letters are from females, denoted FF. Therefore, the Sample Space is $S = \{MM, MF, FM, FF\}$. All outcomes in the sample space are equally likely to occur. We want to find the probability that both the letters are from females given that at least one of them is from a female.

$$\Pr(\{FF\} \mid \{MF, FM, FF\}) = \frac{\Pr(\{FF\})}{\Pr(\{MF, FM, FF\})} = \frac{1}{3}$$

■

Exercise 3.46 On an average, a waiter gets no tip from two of his customers on Saturdays. What is the probability that on next Saturday, he will get no tip from three of his customers?

(a) $\frac{9}{2}e^{-3}$
(b) $2e^{-3}$
(c) $\frac{4}{3}e^{-2}$
(d) $3e^{-2}$

A 3.46

(c) $\frac{4}{3}e^{-2}$

Let X denote the number of customers who will not tip the waiter next saturday. It is given that $\mathbb{E}(X) = 2$ and $X \sim \text{Pois}(2)$. Therefore

$$\Pr(X = 3) = \frac{2^3}{3!}e^{-2} = \frac{4}{3}e^{-2}$$

Exercise 3.47 A linear regression model $y = \alpha + \beta x + \varepsilon$ is estimated using OLS. It turns out that the estimated R^2 equals zero. This implies that:

(a) All x's are necessarily zero
(b) $\hat{\beta} = 1$ and $y = \hat{\alpha} + x$
(c) $\hat{\beta} = 0$ or all x's are constant
(d) There are no implications for $\hat{\beta}$

A 3.47

(c) $\hat{\beta} = 0$ or all x's are constant

Result follows from the fact that $R^2 = \hat{\beta}^2 \frac{\sum(x_i - \bar{x})^2}{\sum(y_i - \bar{y})^2}$.

Exercise 3.48 Using ordinary least squares, a market analyst estimates the following demand function

$$\log X = \alpha + \beta \log P + \varepsilon$$

where X is the output and P is the price. In another formulation, she estimates the above function after dividing all prices by 1000. Comparing the two sets of estimates she would find that

(a) $\hat{\alpha}$ and $\hat{\beta}$ will be the same in both formulations
(b) $\hat{\alpha}$ and $\hat{\beta}$ will differ across both formulations
(c) $\hat{\alpha}$ will change but $\hat{\beta}$ will not
(d) $\hat{\beta}$ will change but $\hat{\alpha}$ will not

A 3.48

(c) $\hat{\alpha}$ will change but $\hat{\beta}$ will not

Let $\hat{\alpha}, \hat{\beta}$ be the estimates of α, β in the following model:

$$\log X = \alpha + \beta \log P + \varepsilon \quad \ldots (i)$$

If we divide the prices by 1000, then the above equation becomes

$$\log X = \alpha - \beta \log(1000) + \beta \log P + \varepsilon \quad \ldots (ii)$$

Clearly both the regressions are the same and yield the same estimate of β. But estimate of α in the latter model is $\hat{\alpha} + \hat{\beta} \log(1000)$.

Exercise 3.49 A linear regression model is estimated using ordinary least squares $y = \alpha + \beta x + \varepsilon$. But the variance of the error term is not constant, and in fact varies directly with another variable z, which is not included in the model. Which of the following statements is true?

(a) The OLS estimated coefficients will be biased because of the correlation between x and the error term
(b) The OLS estimated coefficients will be unbiased but their estimated standard errors will be biased
(c) The OLS estimated coefficients will be unbiased and so will their estimated standard errors because the error variance is not related to x, but to z which is not included in the model
(d) Both the OLS estimated coefficients and their estimated standard errors will be biased

A 3.49

(c) The OLS estimated coefficients will be unbiased and so will their estimated standard errors because the error variance is not related to x, but to z which is not included in the model

A linear regression model is estimated using ordinary least squares, $y = \alpha + \beta x + \varepsilon$. Since variance of the error term is related to a variable that is not part of the model, therefore it will not have any impact on the quality of the estimates and the standard error in the present model.

Exercise 3.50 Suppose the distribution function $F(x)$ of a random variable X is rising in the interval $[a,b)$ and horizontal in the interval $[b,c]$. Which of the following statements CAN be correct?

(a) $F(x)$ is the distribution function of a continuous random variable X.
(b) c is not the largest value that X can take.
(c) The probability that $X = b$ is strictly positive.
(d) Any of the above

A 3.50

(d) Any of the above

Consider the following distribution functions:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{y}{2} & \text{if } 0 \leq y < 1 \\ 0.75 & \text{if } 1 \leq y < 2 \\ 1 & \text{if } y \geq 1 \end{cases}$$

In each the above distribution functions, consider $a = 0$, $b = 1$ and $c = 1.5$. Both the distribution functions (of random variables X and Y) are rising in the interval $[0,1)$ and constant in the interval $[1,1.5]$. $F_X(x)$ is the distribution function of a continuous random variable X. $c = 1.5$ is not the largest value that Y can take. The probability that $Y = b(= 1)$ is strictly positive.

Answer 51, 52 and 53 based on the following information:
Consider an open economy simple Keynesian model with autonomous investment (I). People save a constant proportion (s) of their disposable income and consume the rest. Government taxes the total income at a constant rate (τ) and spends an exogenous amount (G) on various administrative activities. The level of export (X) is autonomous at the fixed real exchange rate (normalized to unity). Import (M) on the other hand is a linear function of total income with a constant import propensity m. Let

$$I = 3200;\ G = 4000;\ X = 800;\ s = \frac{1}{2};\ \tau = \frac{2}{5};\ \text{and } m = \frac{1}{10}$$

Exercise 3.51 The equilibrium level of income is given by:
(a) 8,000
(b) 16,000
(c) 10,000
(d) None of the above

A 3.51

(c) 10,000

We can use the following goods' market equilibrium condition to solve for the equilibrium Y:

$$Y = C(Y_d) + I + G + X - M(Y)$$

where $C(Y_d)$ is the consumption as a function of disposable income Y_d and Y denotes the income. Using the information available in the problem, we can rewrite the above condition as

$$Y = (1-s)(1-\tau)Y + I + G + X - mY$$

equivalently,

$$Y = \frac{I+G+X}{(1-(1-s)(1-\tau)+m)}$$

Using the data from the problem, we get

$$Y = \frac{8000}{(8/10)} = 10000$$

Exercise 3.52 At this equilibrium level of income

(a) there is trade surplus
(b) there is a trade deficit
(c) trade is balanced
(d) one cannot comment on the trade account without further information

A 3.52

(b) there is a trade deficit

Equilibrium income, $Y = 10000$
Exports, $X = 800$
Imports, $M = \frac{1}{10} \times 10000 = 1000$
Therefore, there is a trade deficit of the amount $M - X = 200$.

Exercise 3.53 Now suppose the government decides to maintain a balanced trade by appropriately adjusting the tax rate τ (thereby affecting domestic absorption) - without changing the exchange rate or the amount of government expenditure. Values of other parameters remain the same. The government

(a) can attain this by decreasing the tax rate to 1/5
(b) can attain this by increasing the tax rate to 4/5
(c) can attain this by simply keeping that tax rate unchanged at 2/5
(d) can never attain this objective by adjusting only the tax rate

A 3.53

(c) can attain this by simply keeping that tax rate unchanged at 2/5

Goods market equilibrium condition is

$$Y = (1-s)(1-\tau)Y + I + G + X - mY$$

If the Government adjust τ to ensure balanced budget then $G = \tau Y$. Replacing τ by G/Y in the above equilibrium condition, we can rewrite it as

$$Y = (1-s)(Y) + I + sG + X - mY$$

or equivalently

$$Y = \frac{I + sG + X}{s + m}$$

Using the data from the problem, we get

$$Y = \frac{6000}{(6/10)} = 10000$$

Therefore, $\tau = \dfrac{G}{Y} = \dfrac{4000}{10000} = \dfrac{2}{5}$.

Exercise 3.54 Suppose in an economy banks maintain a cash reserve ratio of 20%. People hold 25% of their money in currency form and the rest in the form of demand deposits. If government increases the high-powered money by 2000 units, the corresponding increase in the money supply would be

(a) 5000 units
(b) 2000 units
(c) 7200 units
(d) None of the above

A 3.54

(a) 5000 units

Let H denotes the high-powered money, C denotes the cash, DD denotes the demand deposits, R be the reserves and M denotes the money supply. Money supply equals cash plus demand deposits.

$$M = C + DD$$

High-powered money equals cash plus reserves.

$$H = C + R$$

We are given that cash reserve ratio is 20%

$$\frac{R}{DD} = 0.2$$

And people hold 25% of their money in currency form and the rest in the form of demand deposits.

$$C = 0.25M; DD = 0.75M$$

Let us first write C and R as a function of DD,

$$C = 0.25(C + DD) \implies C = \frac{1}{3}DD$$

$$R = \tfrac{1}{5}DD$$

Therefore,

$$H = C + R = \frac{8}{15}DD; \quad M = C + DD = \frac{4}{3}DD = \frac{4}{3}\left(\frac{15}{8}H\right) = 2.5H$$

Thus, increase in H by 2000 units, increases M by $2.5 \times 2000 (= 5000)$ units.

Exercise 3.55 Consider the standard IS-LM framework with exogenous money supply. Now suppose the government introduces an endogenous money supply rule such that the money supply becomes an increasing function of the interest rate. As compared to the standard IS-LM case, now

(a) the IS curve will be flatter and fiscal policy would be more effective
(b) the IS curve will be steeper and fiscal policy would be less effective
(c) the LM curve will be flatter and fiscal policy would be more effective
(d) the LM curve will be steeper and fiscal policy would be less effective

A 3.55

(c) the LM curve will be flatter and fiscal policy would be more effective

Let the IS and LM curve be

$$Y = C(Y) + I(r) + G; \quad M = T(Y) + S(r)$$

where $I_r < 0$ and $0 < C_Y < 1$, $T_Y > 0$ and $S_r < 0$

Slope of the LM curve in this case is $\frac{T_Y}{|S_r|}$.

To see the effect of fiscal policy, we differentiate IS and LM *wrt* G

$$(1 - C_Y)\frac{dY}{dG} = I_r\frac{dr}{dG} + 1; \quad 0 = T_Y\frac{dY}{dG} + S_r\frac{dr}{dG}$$

and solve for $\frac{dY}{dG}$,

$$\frac{dY}{dG} = \frac{S_r}{(1 - C_Y)S_r + T_Y I_r}$$

When we introduce endogenous money supply such that money supply increases with interest rate, we get LM curve as

$$M(r) = T(Y) + S(r)$$

Slope of the LM curve in this case is $\frac{T_Y}{|S_r| + M_r} < \frac{T_Y}{|S_r|}$. Therefore, new LM is flatter. Solving for $\frac{dY}{dG}$ in the new scenario, we will get

$$\frac{dY}{dG} = \frac{S_r - M_r}{(1 - C_Y)(S_r - M_r) + T_Y I_r} > \frac{S_r}{(1 - C_Y)(S_r) + T_Y I_r}$$

Therefore, fiscal policy is more efective.

Answer 56, 57, 58, 59 and 60 based on the following information:
Consider an economy where aggregate output is produced by using two factors: capital (K) and labour (L). Aggregate production technology is given by the following production function:

$$Y_t = \alpha K_t + \beta L_t, \quad \text{where } \alpha, \beta > 0$$

At every point of time both factors are fully employed; each worker is paid a wage rate β and each unit of capital is paid a rental price α. A constant proportion s of total output is saved and invested in every period - which augments the capital stock in the next period (no depreciation of capital). Labour force grows at a constant rate n.

Exercise 3.56 This production function violates
(a) the neoclassical property of constant returns to scale
(b) the neoclassical property of diminishing returns to each factor
(c) the neoclassical property of factor returns being equal to the respective marginal products
(d) the neoclassical property of substitutability between capital and labour

A 3.56

(b) the neoclassical property of diminishing returns to each factor

Diminishing Marginal Products is one of the neoclassical property. For the production function in question $F(K,L) = \alpha K + \beta L$, this property is violated because $\dfrac{\partial^2 F}{(\partial K)^2} = 0$.

Exercise 3.57 Let $k_t = K_t/L_t$. The corresponding *per capita* output, y_t, is given by which of the following equations?

(a) $y_t = \alpha + \beta k_t$
(b) $y_t = \beta(k_t)^\alpha$
(c) $y_t = \alpha k_t + \beta$
(d) $y_t = \alpha(k_t)^\beta$

A 3.57

(c) $y_t = \alpha k_t + \beta$

Dividing the production function by L_t we get,

$$\begin{aligned} \frac{Y_t}{L_t} &= \alpha\frac{K_t}{L_t} + \beta\frac{L_t}{L_t} \\ y_t &= \alpha k_t + \beta \end{aligned}$$

Exercise 3.58 The dynamic equation for capital accumulation per worker is given by

(a) $\frac{dk}{dt} = s\beta k_t^\alpha - nk_t$
(b) $\frac{dk}{dt} = s\alpha k_t + s\beta - nk_t$
(c) $\frac{dk}{dt} = s\alpha k_t - nk_t$
(d) $\frac{dk}{dt} = s\alpha k_t^\beta - nk_t$

A 3.58

(b) $\frac{dk}{dt} = s\alpha k_t + s\beta - nk_t$

Dynamic Equation for capital accumulation is

$$\frac{dK_t}{dt} = s(\alpha K_t + \beta L_t) \cdots (*)$$

To find the Dynamic Equation for capital accumulation per worker, we use the following:

$$k_t \equiv \frac{K_t}{L_t}$$

Therefore,

$$\log k_t \equiv \log K_t - \log L_t$$

Differentiating with respect to t,

$$\frac{1}{k_t}\frac{dk_t}{dt} = \frac{1}{K_t}\frac{dK_t}{dt} - n$$

implying

$$\frac{dK_t}{dt} = L_t\frac{dk_t}{dt} + nK_t$$

Substituing in $(*)$, we get

$$L_t\frac{dk_t}{dt} = s(\alpha K_t + \beta L_t) - nK_t$$

And hence the result:

$$\frac{dk_t}{dt} = s\alpha k_t + s\beta - nk_t$$

Exercise 3.59 Let $\alpha = \frac{1}{2}$; $\beta = 12$; $s = \frac{1}{4}$; $n = \frac{1}{2}$. The corresponding steady state value of capital per worker is given by

(a) 8
(b) 36
(c) $(4)^{\frac{1}{11}}$
(d) There does not exist any well-defined steady state value

A 3.59

(a) 8

In the steady state, $\frac{dk_t}{dt} = 0$. Therefore, solving

$$\frac{dk_t}{dt} = s\alpha k_t + s\beta - nk_t = 0$$

and substituting values of $\alpha = \frac{1}{2}$; $\beta = 12$; $s = \frac{1}{4}$; $n = \frac{1}{2}$, we get

$$k_t^* = \frac{s\beta}{n - s\alpha} = 8$$

Exercise 3.60 Consider the same set of parameter values as above. An increase in the savings ratio

(a) unambiguously increases the steady state value of capital per worker
(b) unambiguously decreases the steady state value of capital per worker
(c) leaves the steady state value of capital per worker unchanged
(d) has an ambiguous effect; a steady state may not exist if the savings ratio increases sufficiently

A 3.60

(a) unambiguously increases the steady state value of capital per worker

In the previous problem, we derived that in the steady state

$$k_t^* = \frac{s\beta}{n - s\alpha}$$

An increase in savings rate (s), unambiguously increases the steady state value of capital per worker (numerator is increasing in s and denominator is decreasing in s).

4. DSE - 2014

Solved Problems - DSE 2014

Exercise 4.1 Suppose that we classify all households into one of two states, rich and poor. The probability of a particular generation being in either of these states depends only on the state in which their parents were. If a parent is poor today, their child is likely to be poor with probability 0.7. If a parent is rich today, their child is likely to be poor with probability 0.6. What is the probability that the great grandson of a poor man will be poor?

(a) 0.72
(b) 0.67
(c) 0.62
(d) 0.78

A 4.1

(b) 0.67

Let E_0 be the event that the man is poor, E_1 be the event that the son of a man is poor, E_2 be the event that the grandson of a man is poor and E_3 be the event that the great grandson of a man is poor. We are asked to find

$$\begin{aligned}
\Pr(E_3|E_0) &= \Pr(E_3 \cap E_2 \cap E_1|E_0) + \Pr(E_3 \cap E_2 \cap E_1^c|E_0) + \\
&\quad \Pr(E_3 \cap E_2^c \cap E_1|E_0) + \Pr(E_3 \cap E_2^c \cap E_1^c|E_0) \\
&= \Pr(E_3|E_2)\Pr(E_2|E_1)\Pr(E_1|E_0) + \Pr(E_3|E_2)\Pr(E_2|E_1^c)\Pr(E_1^c|E_0) + \\
&\quad \Pr(E_3|E_2^c)\Pr(E_2^c|E_1)\Pr(E_1|E_0) + \Pr(E_3|E_2^c)\Pr(E_2^c|E_1^c)\Pr(E_1^c|E_0) \\
&= 0.7 \times 0.7 \times 0.7 + 0.7 \times 0.6 \times 0.3 + 0.6 \times 0.3 \times 0.7 + 0.6 \times 0.4 \times 0.3 \\
&= 0.667
\end{aligned}$$

Exercise 4.2 Consider the experiment of tossing two fair coins. Let the event A be a head on the first coin, the event C be a head on the second coin, the event D be that both coins match and the event G be two heads. Which of the following is *false*?

(a) C and D are statistically independent
(b) A and G are statistically independent
(c) A and D are statistically independent
(d) A and C are statistically independent

A 4.2

(b) A and G are statistically independent

$\Pr(A) = 0.5, \Pr(G) = 0.25$
$\Pr(A \cap G) = \Pr(G) = 0.25 \neq 0.125 = \Pr(A) \times \Pr(G)$
Thus, A and G are not independent.

Exercise 4.3 Let Y denote the number of heads obtained when 3 fair coins are tossed. Then the expectation of $Z = 4 + 5Y^2$ is

(a) 17
(b) 18
(c) 19
(d) None of the above

A 4.3

(c) 19

Note that $Y \sim B(3, 0.5)$ i.e. Y is a binomial random variable with parameters $n = 3$ and $p = 0.5$. Thus, $\mathbb{E}(Y) = np = 1.5$ and $\mathbb{V}(Y) = np(1-p) = 0.75$. Therefore,

$$\begin{aligned}\mathbb{E}(Z) &= \mathbb{E}(4+5Y^2)\\ &= 4+5\mathbb{E}(Y^2)\\ &= 4+5\mathbb{V}(Y)+5(\mathbb{E}(Y))^2\\ &= 4+5(0.75)+5(1.5)^2\\ &= 19\end{aligned}$$

Exercise 4.4 Let Y denote the number of heads obtained when 3 fair coins are tossed. Then the variance of $Z = 4 + 5Y^2$ is

(a) 185.5
(b) 178.5
(c) 187.5
(d) None of the above

A 4.4

(c) 187.5

$$\begin{aligned}\mathbb{V}(Z) &= 25\mathbb{V}(Y^2)\\ &= 25(\mathbb{E}(Y^4)-(\mathbb{E}(Y^2))^2)\\ &= 25(\mathbb{E}(Y^4)-9)\\ &= 25(1(3)(0.125)+16(3)(0.125)+81(0.125)-9)\\ &= 187.5\end{aligned}$$

Exercise 4.5 Let events E, F and G be pairwise independent with $\Pr(G) > 0$ and $\Pr(E \cap F \cap G) = 0$. Let X^c denote the complement of event X. Then $\Pr(E^c \cap F^c | G) =$

(a) $\Pr(E^c) + \Pr(F^c)$
(b) $\Pr(E^c) - \Pr(F^c)$
(c) $\Pr(E^c) - \Pr(F)$
(d) None of the above

A 4.5

(c) $\Pr(E^c) - \Pr(F)$

$$\begin{aligned}\Pr(E^c \cap F^c | G) &= \frac{\Pr(E^c \cap F^c \cap G)}{\Pr(G)}\\ &= \frac{\Pr(G) - \Pr(E \cap G) - \Pr(F \cap G)}{\Pr(G)}\\ &= \frac{\Pr(G) - \Pr(E)\Pr(G) - \Pr(F)\Pr(G)}{\Pr(G)}\\ &= 1 - \Pr(E) - \Pr(F)\\ &= \Pr(E^c) - \Pr(F)\end{aligned}$$

Exercise 4.6 Let $a_n = \left(1 + \frac{1}{n}\right)^{n+1}$, $n = 1, 2, \ldots$ Then the sequence $(a_n)_{n=1}^{\infty}$

(a) is an increasing sequence
(b) first increases, then decreases
(c) is a decreasing sequence
(d) first decreases, then increases

A 4.6

(c) is a decreasing sequence

Given the sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \left(1+\frac{1}{n}\right)^{n+1}$$

We will consider its monotonic transformation by taking log both sides

$$\begin{aligned}
\log a_n &= (n+1)\log\left(1+\frac{1}{n}\right) \\
&= (n+1)\log\left(\frac{n+1}{n}\right) \\
&= (n+1)(\log(n+1)-\log(n))
\end{aligned}$$

Differentiating above with respect to n, we get

$$\begin{aligned}
\frac{d(\log a_n)}{dn} &= (n+1)\left(\frac{1}{n+1}-\frac{1}{n}\right)+(\log(n+1)-\log(n)) \\
&= \left(-\frac{1}{n}\right)+(\log(n+1)-\log(n)) \\
&< 0
\end{aligned}$$

The above inequality holds because $f(x)=\log x$ is a strictly concave function and $f'(x) > f(x+1)-f(x)$ for all x when f is strictly concave. Therefore, (a_n) is a decreasing sequence.

Exercise 4.7 Let M, A, B, C be respectively the four matrices below:

$$\begin{pmatrix}4&7\\7&9\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}1&2\\3&4\end{pmatrix},\begin{pmatrix}1&1\\4&5\end{pmatrix}$$

Then $M = xA+yB+zY$,

(a) but x, y, z are not unique.

(b) $z=-1$

(c) $z=-1$ and $z=-2$ both can hold

(d) x, y, z are unique but $z=2$

A 4.7

(b) $z = -1$

Clearly, $x = 2, y = 3, z = -1$ is the only solution to the following system of equations

$$\begin{aligned} x+y+z &= 4 \\ x+2y+z &= 7 \\ x+3y+4z &= 7 \\ x+4y+5z &= 9 \end{aligned}$$

Therefore, $z = -1$.

Exercise 4.8 Let f be a continuous function from $[a,b]$ to $[a,b]$, and is differentiable on (a,b). We will say that point $y \in [a,b]$ is a *fixed point* of f if $y = f(y)$. If the derivative $f'(x) \neq 1$ for any $x \in (a,b)$, then f has

(a) multiple, and an odd number of, fixed points
(b) no fixed points in $[a,b]$
(c) multiple, but an even number of, fixed points
(d) exactly one fixed point in $[a,b]$

A 4.8

(d) exactly one fixed point in $[a,b]$

First we will show that f has at least one fixed point and then we will show that it cannot have more than one fixed point. To show the former, define a function g from $[a,b]$ to $\mathbb{R}$ in the following way:

$$g(x) = f(x) - x$$

g is continuous because f is continuous. Also, $g(a) \geq 0$ and $g(b) \leq 0$. Therefore, by intermediate value theorem, there exists $y_1 \in [a,b]$ such that $g(y_1) = f(y_1) - y_1 = 0$. This shows that the fixed point y_1 of f exists. To show the uniqueness we suppose that there are more than one fixed points. Let y_2 be the another fixed point. So, $y_1 = f(y_1)$ and $y_2 = f(y_2)$ and without loss of generality, lets say $y_1 < y_2$. By Mean Value Theorem, there exists $c \in (y_1, y_2) \subset (a,b)$ such that $f'(c) = 1$. This results in a contradiction as we have $f'(x) \neq 1$ for any $x \in (a,b)$.

Exercise 4.9 Which of the following statements is true for all real numbers a, b with $a < b$?

(a) $\sin b - \sin a \leq b - a$
(b) $\sin b - \sin a \geq b - a$
(c) $|\sin b - \sin a| \geq b - a$
(d) $|\sin b - \sin a| \leq |b - a|$

A 4.9

(d) $|\sin b - \sin a| \leq |b-a|$

The result follows from the fact that $\left|\dfrac{d\sin x}{dx}\right| = |\cos x| \leq 1$. Additionally, we can also rule out option (a) by taking $b=0$ and $a=\pi$ and rule out options (b) and (c) by taking $b=\pi$ and $a=0$.

Exercise 4.10 Let $O(0,0)$, $P(3,4)$ and $Q(6,0)$ be the vertices of a triangle OPQ. If a point S in the interior of the OPQ is such that triangles OPS, PQS and OQS have equal area, then the coordinates of S are:

(a) $(4/3,3)$
(b) $(3,2/3)$
(c) $(3,4/3)$
(d) $(4/3,2/3)$

A 4.10

(c) $(3,4/3)$

We can do this problem quickly by working with options. Area of the triangle OPQ is $(1/2)(6)(4)=12$. So, area of triangle OQS must be 4. If S is $(4/3,3)$, area of OQS is $(1/2)(6)(3)=9\neq 4$. If S is $(3,2/3)$, area of OQS is $(1/2)(6)(2/3)=2\neq 4$. If S is $(4/3,2/3)$, area of OQS is $(1/2)(6)(2/3)=2\neq 4$. If S is $(3,4/3)$, area of OQS is $(1/2)(6)(4/3)=4$. Thus eliminating options (a), (b) and (d), we get (c) as the correct answer.

Exercise 4.11 5 men and 5 women are seated randomly in a single row of chairs. The expected number of women sitting next to at least one man equals

(a) $11/3$
(b) $13/3$
(c) $35/9$
(d) $37/9$

A 4.11

(c) $35/9$

Let the seats be numbered $1,2,3,\dots,10$ in order of arrangement. Define a random variable I_1 in the following way: I_1 takes value 1 if seat number 1 is occupied by a woman and seat number 2 is occupied by a man, and 0 otherwise. Likewise, we define random variables I_j for $2 \le j \le 9$ in the similar way: I_j takes value 1 if seat number j is occupied by a woman and at least one of the two seats numbered $(j-1)$ and $(j+1)$ is occupied by a man, and takes value 0 otherwise. I_{10} takes value 1 if seat number 10 is occupied by a woman and seat number 9 is occupied by a man, and 0 otherwise. Let N be the number of women seating next to at least one man. Therefore,

$$N = I_1 + I_2 + \cdots + I_9 + I_{10}$$

By linearity of expectation,

$$\mathbb{E}(N) = \mathbb{E}(I_1) + \mathbb{E}(I_2) + \cdots + \mathbb{E}(I_9) + \mathbb{E}(I_{10})$$

By symmetry,

$$\mathbb{E}(I_1) = \mathbb{E}(I_{10}) \quad \text{and} \quad \mathbb{E}(I_2) = \cdots = \mathbb{E}(I_9)$$

So, we just need to find $\mathbb{E}(I_1)$ and $\mathbb{E}(I_2)$.

$$\begin{aligned}
\mathbb{E}(I_1) &= \Pr(I_1 = 1) = \frac{\binom{5}{1}\binom{5}{1}8!}{10!} = \frac{5}{18} \\
\mathbb{E}(I_2) &= \Pr(I_2 = 1) \\
&= 1 - \Pr(I_2 = 0) \\
&= 1 - [\Pr(\text{seat number 2 is occupied by man}) + \\
&\qquad \Pr(\text{seat numbers 1, 2 and 3 are occupied by women})] \\
&= 1 - \left[\frac{\binom{5}{1}9!}{10!} + \frac{\binom{5}{3}3!7!}{10!}\right] = 1 - \left[\frac{1}{2} + \frac{1}{12}\right] = \frac{5}{12}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(N) &= \mathbb{E}(I_1) + \mathbb{E}(I_2) + \cdots + \mathbb{E}(I_9) + \mathbb{E}(I_{10}) \\
&= [2 \times \mathbb{E}(I_1)] + [8 \times \mathbb{E}(I_2)] = \frac{5}{9} + \frac{10}{3} = \frac{35}{9}
\end{aligned}$$

Exercise 4.12 Let M be a 3×3 matrix such that $M^2 = M$. Which of the following is necessarily true?

(a) M is invertible
(b) $\det(M) = 0$
(c) $\det(M^5) = \det(M)$
(d) None of the above

A 4.12

(c) $\det(M^5) = \det(M)$

$$\begin{aligned} M^5 &= M^2 \times M \times M^2 \\ &= M \times M \times M \\ &= M \times M^2 \\ &= M \times M \\ &= M^2 \\ &= M \end{aligned}$$

Thus, $\det(M^5) = \det(M)$.

Exercise 4.13 Suppose a straight line in $\mathbb{R}^3$ passes through the point $(-1,3,3)$ in the direction of the vector $(1,2,3)$. The line will intersect with the xy-plane at point
(a) $(2,-1,0)$
(b) $(1,3,0)$
(c) $(3,1,0)$
(d) None of the above

A 4.13

(d) None of the above

Both the points $(-1,3,3)$ and $(1,2,3)$ lie in the plane $z = 3$. A line through these two points on the plane $z = 3$ will stay in the plane and will never intersect a parallel plane $z = 0$ i.e. xy-plane.

Exercise 4.14 X is a random variable. Which of the following statements is always true?
(a) The expectation of X exists.
(b) The distribution function of X is strictly increasing
(c) X has a median
(d) None of the above

A 4.14

(c) X has a median

m is known as the Median of X if $\Pr(X \geq m) \geq 0.5$ and $\Pr(X \leq m) \geq 0.5$. We will show that it always exists. Define $m_0 := \sup\{a | \Pr(X \geq a) \geq \frac{1}{2}\}$. Note that $\{a | \Pr(X \geq a) \geq \frac{1}{2}\}$ is an interval that is bounded above. So, m_0 exists. We will now show that m_0 is the median of X. For $a > m_0$, from the definition of m_0, $\Pr(X \geq a) < \frac{1}{2}$. Now since $\lim_{a \to m_0^+} \Pr(X \geq a)$ exists and equals $\Pr(X > m_0)$, it follows that $\Pr(X > m_0) \leq \frac{1}{2}$. That is, $\Pr(X \leq m_0) \geq \frac{1}{2}$. For the other direction, consider any $a < m_0$, we have $\Pr(X \geq a) \geq \frac{1}{2}$. Now since $\lim_{a \to m_0^-} \Pr(X \geq a) = \Pr(X \geq m_0)$, it follows that $\Pr(X \geq m_0) \geq \frac{1}{2}$.

Exercise 4.15 Consider two disjoint events A and B in a sample space S. Which of the following is correct?

(a) A and B are always independent
(b) A and B cannot be independent
(c) A and B are independent if both of them have positive probability
(d) None of the above

A 4.15

(d) None of the above

Consider any A such that $0 < \Pr(A) < 1$. Clearly A and A^c are both disjoint and have positive probability but they are not independent. This rules out options (a) and (c). If we consider A as before and $B = \emptyset$, then A and B are disjoint and independent. This rules out option (b).

The following information is the starting point for Q 16 - 20. Consider an exchange economy with two agents, 1 and 2, and two goods, X and Y. Agent 1's endowment is $(0, 10)$ and agent 2's endowment is $(11, 0)$. Agent 1 strictly prefers bundle (a, b) to bundle (c, d) if, either $a > c$, or $a = c$ and $b > d$. Agent 2 strictly prefers bundle (a, b) to bundle (c, d) if $\min\{a, b\} > \min\{c, d\}$. For both agents, we say that bundle (a, b) is indifferent to bundle (c, d) if, neither (a, b), nor (c, d), is strictly preferred to the other.

Exercise 4.16 This exchange economy has
(a) one competitive equilibrium allocation
(b) two competitive equilibrium allocations
(c) an infinite number of competitive equilibrium allocations
(d) no competitive equilibrium allocations

A 4.16

(a) One competitive equilibrium allocation

Demand for commodity X by agent 1 is given by

$$x_1(p_x, p_y, m_1) = \frac{m_1}{p_x}$$

where p_x denotes price of X, p_y denotes price of Y and m_1 is income of agent 1.
Demand for commodity X by agent 2 is given by

$$x_2(p_x, p_y, m_2) = \frac{m_2}{p_x + p_y}$$

where m_2 is income of agent 2. We will now replace $m_1 = 10p_y$ and $m_2 = 11p_x$ by the value of the endowments of agents 1 and 2 respectively and take $p_x = 1$ as numeraire price. So, demands for X by 1 and 2 can now be written as

$$x_1 = \frac{10p_y}{1} \quad x_2 = \frac{11}{1+p_y}$$

We can now solve for p_y by equating demand and supply

$$\frac{10p_y}{1} + \frac{11}{1+p_y} = 11$$

gives us

$$p_y = \frac{1}{10}$$

Therefore, competitive equilibrium allocation is $((x_1, y_1), (x_2, y_2)) = ((1, 0), (10, 10))$

■

Exercise 4.17 Which of the following changes makes $(p_x, p_y) = (1, 0)$ a competitive equilibrium price vector?

(a) agent 2's endowment changes to $(9, 0)$
(b) agent 2's endowment changes to $(10, 0)$
(c) agent 1's endowment changes to $(0, 12)$
(d) none of the above

A 4.17

(d) none of the above

When $p_y = 0$, agent 1's demand for commodity Y is not finite. So, market can never clear regardless of what the endowment is.

■

Exercise 4.18 Suppose only agent 2's preferences are changed. The changed preferences of agent 2's become identical to those of agent 1. Then,

(a) there is no equilibrium price ratio
(b) both of the following are true
(c) $p_x/p_y = 0$ is an equilibrium price ratio
(d) $p_y/p_x = 0$ is an equilibrium price ratio

A 4.18

(a) there is no equilibrium price ratio

When $p_y = 0$, both agent 1's and 2's demand for commodity Y is not finite. This rules out (d) and therefore (b). When $p_x = 0$, both agent 1's and 2's demand for commodity X is not finite. This rules out (c). When both prices are positive, then demand for Y is always 0, so market can never clear. This shows that there is no equilibrium price ratio.

Exercise 4.19 Suppose only agent 2's preferences are changed. The changed preferences is such that agent 2 strictly prefers bundle (a,b) to bundle (c,d) if, either $b > d$, or $b = d$ and $a > c$. Then,

(a) there is no equilibrium price ratio
(b) both of the following are true
(c) $p_x/p_y = 0$ is an equilibrium price ratio
(d) $p_x/p_y > 0$ is an equilibrium price ratio

A 4.19

(e) $p_x/p_y = 10/11$ is an equilibrium price ratio

Clearly, none of the prices can be zero because zero price of any commodiy would result in infinite demand for it. So, we will only consider positive prices. Demand for commodity X by agent 1 is given by

$$x_1(p_x, p_y, m_1) = \frac{m_1}{p_x}$$

where p_x denotes price of X, p_y denotes price of Y and m_1 is income of agent 1.
Demand for commodity X by agent 2 is given by

$$x_2(p_x, p_y, m_2) = 0$$

where m_2 is income of agent 2. We will now replace $m_1 = 10p_y$ and $m_2 = 11p_x$ by the value of the endowments of agents 1 and 2 respectively. So, demands for X by 1 and 2 can now be written as

$$x_1 = \frac{10p_y}{p_x} \quad x_2 = 0$$

We can now solve for equilibrium price ratio $\frac{p_x}{p_y}$ by equating demand and supply

$$\frac{10p_y}{p_x} + 0 = 11$$

gives us

$$\frac{p_x}{p_y} = \frac{10}{11}$$

Therefore, competitive equilibrium allocation is $((x_1, y_1), (x_2, y_2)) = ((11, 0), (0, 10))$

Exercise 4.20 Suppose only agent 1's preferences are changed. The changed preferences of agent 1's become identical to those of agent 2. Then,

(a) there is no equilibrium price ratio
(b) both of the following are true
(c) $p_x/p_y = 0$ is an equilibrium price ratio
(d) $p_y/p_x = 0$ is an equilibrium price ratio

A 4.20

(c) $p_x/p_y = 0$ is an equilibrium price ratio

If we consider the case of $p_x > 0$ and $p_y > 0$ first, demand for commodity X by agent 1 is given by

$$x_1(p_x, p_y, m_1) = \frac{m_1}{p_x + p_y}$$

where p_x denotes price of X, p_y denotes price of Y and m_1 is income of agent 1.
Demand for commodity X by agent 2 is given by

$$x_2(p_x, p_y, m_2) = \frac{m_2}{p_x + p_y}$$

where m_2 is income of agent 2. We will now replace $m_1 = 10p_y$ and $m_2 = 11p_x$ by the value of the endowments of agents 1 and 2 respectively. So, demands for X by 1 and 2 can now be written as

$$x_1 = \frac{10p_y}{p_x + p_y} \quad x_2 = \frac{11p_x}{p_x + p_y}$$

Summing the demands we get the following inequality

$$\frac{11p_x + 10p_y}{p_x + p_y} < \frac{11p_x + 11p_y}{p_x + p_y} = 11$$

Thus, there is an excess supply of X at positive prices. Now considering $p_x = 0$ and $p_y = 1$, we get the set of demands for X by agent 1 as $[10, \infty)$ and by agent 2 as $[0, \infty)$. Therefore, the set of competitive equilibria is

$$\{((x_1, y_1), (x_2, y_2)) | y_1 = 10, y_2 = 0, x_1 + x_2 = 11, 10 \leq x_1 \leq 11\}$$

■

Exercise 4.21 5 men and 5 women are seated randomly in a single circle of chairs. The expected number of women sitting next to at least 1 man equals

(a) $23/6$
(b) $25/6$
(c) 4
(d) $17/4$

A 4.21

(b) $25/6$

Let the seats be numbered $1,2,3,\ldots,10$ in order of arrangement. We define random variables I_j for $1 \leq j \leq 10$ in the following way: I_j takes value 1 if seat number j is occupied by a woman and at least one of the two seats in the neighbourhood is occupied by a man, and takes value 0 otherwise. Let N be the number of women seating next to at least one man. Therefore,

$$N = I_1 + I_2 + \cdots + I_9 + I_{10}$$

By linearity of expectation,

$$\mathbb{E}(N) = \mathbb{E}(I_1) + \mathbb{E}(I_2) + \cdots + \mathbb{E}(I_9) + \mathbb{E}(I_{10})$$

By symmetry,

$$\mathbb{E}(I_1) = \mathbb{E}(I_2) = \cdots = \mathbb{E}(I_{10})$$

So, we just need to find $\mathbb{E}(I_1)$

$$\begin{aligned}
\mathbb{E}(I_1) &= \Pr(I_1 = 1) \\
&= 1 - \Pr(I_1 = 0) \\
&= 1 - [\Pr(\text{seat number 1 is occupied by man}) + \\
&\quad \Pr(\text{seat numbers 10, 1 and 2 are occupied by women})] \\
&= 1 - \left[\frac{\binom{5}{1}9!}{10!} + \frac{\binom{5}{3}3!7!}{10!}\right] = 1 - \left[\frac{1}{2} + \frac{1}{12}\right] = \frac{5}{12}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(N) &= 10\mathbb{E}(I_1) \\
&= \frac{25}{6}
\end{aligned}$$

■

Exercise 4.22 Ms. A selects a number X randomly from the uniform distribution on $[0,1]$. Then Mr. B repeatedly, and independently, draws numbers $Y_1, Y_2, \ldots$ from the uniform distribution on $[0,1]$, until he gets a number larger than $X/2$, then stops. The expected number of draws that Mr. B makes equals

(a) $2\ln 2$
(b) $\ln 2$
(c) $2/e$
(d) $6/e$

A 4.22

(a) $2\ln 2$

Define

$$N = \inf\left\{n \in \mathbb{N} | Y_n > \frac{X}{2}\right\}$$

We want to determine the value of $\mathbb{E}(N)$. By law of iterated expectations,

$$\mathbb{E}(N) = \mathbb{E}(\mathbb{E}(N|X))$$

$N|X = x \sim \text{Geom}\left(1 - \frac{x}{2}\right)$ i.e. $N|X = x$ is the number of draws until first success where success is getting a draw value more than $x/2$. Since Y_is are i.i.d draws from $\mathcal{U}(0,1)$, the probability of success in each trial is $\left(1 - \frac{x}{2}\right)$ and $N|X = x$ is geometric. Therefore,

$$\mathbb{E}(N|X = x) = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2 - x}$$

Hence, we get

$$\mathbb{E}(N) = \mathbb{E}(\mathbb{E}(N|X)) = \int_0^1 \mathbb{E}(N|X = x) f_X(x) dx = \int_0^1 \frac{2}{2 - x} dx = 2\ln 2$$

■

Exercise 4.23 Ms. A selects a number X randomly from the uniform distribution on $[0,1]$. Then Mr. B repeatedly, and independently, draws numbers $Y_1, Y_2, \ldots$ from the uniform distribution on $[0,1]$, until he gets a number larger than $X/2$, then stops. The expected sum of the number Mr. B draws, given $X = x$, equals

(a) $\ln 2$
(b) $1/(1 - \frac{1}{2})$
(c) $1/(2 - x)$
(d) $3/(1 - \frac{1}{2})$

A 4.23

(c) $1/(2-x)$

Define

$$N = \inf\left\{n \in \mathbb{N} | Y_n > \frac{X}{2}\right\}$$

and

$$S = \sum_{i=1}^{N} Y_i$$

We want to determine the value of $\mathbb{E}(S|X=x)$. By law of iterated expectations,

$$\begin{aligned}
\mathbb{E}(S|X=x) &= \mathbb{E}(\mathbb{E}(S|N, X=x)|X=x) \\
&= \mathbb{E}(\mathbb{E}(Y_1+Y_2+\cdots+Y_N|N, X=x)|X=x) \\
&= \mathbb{E}([\mathbb{E}(Y_1|N, X=x)+\cdots+\mathbb{E}(Y_N|N, X=x)]|X=x) \\
&= \mathbb{E}\left((N-1)\frac{x}{4}+\frac{2+x}{4}\middle| X=x\right) \\
&= \frac{2+x}{4}+\frac{x}{4}\mathbb{E}((N-1)|X=x)
\end{aligned}$$

$N|X=x \sim \text{Geom}\left(1-\frac{x}{2}\right)$. Therefore, $\mathbb{E}((N-1)|X=x) = \frac{x}{2-x}$ and we get,

$$\begin{aligned}
\mathbb{E}(S|X=x) &= \frac{2+x}{4}+\frac{x^2}{4(2-x)} \\
&= \frac{1}{2-x}
\end{aligned}$$

Exercise 4.24 There are two fair coins (i.e. Heads and Tails are equally likely for tosses of both). Coin 1 is tossed 3 times. Let X be the number of Heads that occur. After this, Coin 2 is tossed X times. Let Y be the number of Heads we get with Coin 2. The probability $\Pr(X \geq 2|Y=1)$ equals

(a) $1/2$
(b) $4/7$
(c) $2/3$
(d) $11/18$

A 4.24

(e) 10/18

$$
\begin{aligned}
\Pr(X \geq 2|Y=1) &= \frac{\Pr(X \geq 2, Y=1)}{\Pr(Y=1)} \\
&= \frac{\Pr(X=2,Y=1)+\Pr(X=3,Y=1)}{\Pr(X=1,Y=1)+\Pr(X=2,Y=1)+\Pr(X=3,Y=1)} \\
&= \frac{\Pr(Y=1|X=2)\Pr(X=2)+\Pr(Y=1|X=3)\Pr(X=3)}{\sum_{i=1}^{3}\Pr(Y=1|X=i)\Pr(X=i)} \\
&= \frac{\left[\frac{1}{2}\times\frac{3}{8}\right]+\left[\frac{3}{8}\times\frac{1}{8}\right]}{\left[\frac{1}{2}\times\frac{3}{8}\right]+\left[\frac{1}{2}\times\frac{3}{8}\right]+\left[\frac{3}{8}\times\frac{1}{8}\right]} \\
&= \frac{5}{9} = \frac{10}{18}
\end{aligned}
$$

Exercise 4.25 Two independent random variables X and Y have the same probability density functions:

$$
f(x) = \begin{cases} c(1+x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}
$$

Then the variance of their sum, $\mathbb{V}(X+Y)$ equals

(a) $2/9$
(b) $13/81$
(c) $4/45$
(d) $5/18$

A 4.25

(b) 13/81

For f to be a valid density $c = 2/3$. Since X and Y are independently and identically distributed, $\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(Y) = 2 \times \mathbb{V}(X) = 2 \times [\mathbb{E}(X^2) - (\mathbb{E}(X))^2]$.

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_0^1 \frac{2}{3}x^2(1+x)dx \\
&= \int_0^1 \frac{2}{3}x^2dx + \int_0^1 \frac{2}{3}x^3dx \\
&= \frac{2}{9} + \frac{1}{6} \\
&= \frac{7}{18}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^1 \frac{2}{3}x(1+x)dx \\
&= \int_0^1 \frac{2}{3}xdx + \int_0^1 \frac{2}{3}x^2dx \\
&= \frac{1}{3} + \frac{2}{9} \\
&= \frac{5}{9}
\end{aligned}$$

Therefore, $\mathbb{V}(X+Y) = 2 \times \left[\frac{7}{18} - \frac{25}{81}\right] = \frac{13}{81}$

Exercise 4.26 Suppose two restaurants are going to be located at a street that is ten kilometres long. The location of each restaurant will be chosen randomly. What is the probability that they will be located less than five kilometres apart?

(a) 1/4
(b) 1/2
(c) 3/4
(d) 1/3

A 4.26

(c) 3/4

Let X be the position of restaurant 1 and Y be the position of restaurant 2. It is given that both X and Y are i.i.d uniformly over $(0,1)$. Thus, the joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

We need to find $\Pr(|X-Y| < 1/2)$.

$$\begin{aligned}
\Pr(|X-Y| < 1/2) &= \Pr(|X-Y| < 1/2, X > Y) + \Pr(|X-Y| < 1/2, X \leq Y) \\
&= \Pr(Y < X < Y + 1/2) + \Pr(X \leq Y < X + 1/2) \\
&= 2 \times \Pr(Y < X < Y + 1/2) \\
&= 2 \times \left[\int_0^{\frac{1}{2}} \int_0^x dydx + \int_{\frac{1}{2}}^1 \int_{x-\frac{1}{2}}^x dydx \right] \\
&= 2 \times \left[\int_0^{\frac{1}{2}} xdx + \int_{\frac{1}{2}}^1 \frac{1}{2} dx \right] \\
&= 2 \times \left[\frac{1}{8} + \frac{1}{4} \right] \\
&= \frac{3}{4}
\end{aligned}$$

Exercise 4.27 Consider the linear regression model: $y_i = \beta_1 D1_i + \beta_2 D2_i + \varepsilon_i$, where $D1_i = 1$ if $1 \leq i \leq N$ and $D1_i = 0$ if $N+1 \leq i \leq n$ for some $1 < N < n$ and $D2_i = 1 - D1_i$. Can this model be estimated using least squares?

(a) No, because $D1$ and $D2$ are perfectly collinear
(b) Yes, and it is equivalent to running two separate regressions of y on $D1$ and y on $D2$, respectively
(c) No, because there is no variability in $D1$ and $D2$
(d) Yes, provided an intercept term is included.

A 4.27

(b) Yes, and it is equivalent to running two separate regressions of y on $D1$ and y on $D2$, respectively

Yes, because there is no intercept term and hence running this regression will not lead to a problem of multi-collinearity. It is equivalent to running two separate regressions y on $D1$ and y on $D2$ because $D1$ and $D2$ are orthogonal.

Exercise 4.28 Consider the least squares regression of y on a single variable x. Which of the following statements is true about such a regression?

(a) The coefficient of determination R^2 is always equal to the squared correlation coefficient between y and x
(b) The coefficient of determination R^2 is equal to the squared correlation coefficient between y and x only if there is no intercept in the equation

(c) The coefficient of determination R^2 is equal to the squared correlation coefficient between y and x only if there is an intercept in the equation

(d) There is no relationship between the coefficient of determination R^2 and the squared correlation coefficient between y and x

A 4.28

(c) The coefficient of determination R^2 is equal to the squared correlation coefficient between y and x only if there is an intercept in the equation

The coefficient of determination R^2 is the ratio of the explained variation to the total variation. The coefficient of determination is equal to the squared correlation coefficient between y and x only if there is an intercept in the equation.

$$
\begin{aligned}
R^2 &= \frac{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2} \\
&= \frac{\sum_{i=1}^{n}(\hat{\alpha}+\hat{\beta}x_i-\hat{\alpha}-\hat{\beta}\bar{x})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2} \\
&= \frac{\sum_{i=1}^{n}(\hat{\beta}x_i-\hat{\beta}\bar{x})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2} \\
&= \frac{\hat{\beta}^2\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2} \\
&= \left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]^2 \times \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2} \\
&= \left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2\sum_{i=1}^{n}(x_i-\bar{x})^2}}\right]^2 \\
&= [\rho]^2
\end{aligned}
$$

The above holds because of the presence of intercept term. In the absence of intercept term, we can't replace $\bar{y}$ by $\hat{\beta}\bar{x}$ since sum of the residuals may not be zero.

Exercise 4.29 An analyst runs two least squares regressions: first, of y on a single variable x, and second of x on y. In both cases, she decides to include an intercept term. Which of the following is true of what she finds?

(a) The slope coefficient of the first regression will be the inverse of the slope coefficient of the second regression; this will also be true of the associated t-ratios

(b) The slope coefficients will be different, the associated t-ratios will also be different, but the R^2 from the two regressions will be the same

(c) The slope coefficients will be different, but the associated t-ratios and the R^2 from the two regressions will be the same

(d) The slope coefficient will be the inverse of each other, the associated t-ratios will also be the inverse of each other, but the R^2 from the two regressions will be the same

A 4.29

(c) The slope coefficients will be different, but the associated t-ratios and the R^2 from the two regressions will be the same

Let $\hat{\beta}_{yx}$ denote the slope coefficient of the regression of y on x and $\hat{\beta}_{xy}$ denote the slope coefficient of the regression of x on y.

$$\hat{\beta}_{yx} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} \neq \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(y_i-\bar{y})^2} = \hat{\beta}_{xy}$$

From Q 28, we know that $R^2 = \rho^2$. Since ρ^2 does not change by swapping x and y, therefore R^2 is the same for the both the regressions. Standard error of the slope coefficient in the regression of y on x is,

$$s_{yx}^2 = \frac{\sum_{i=1}^{n} e_i^2}{(n-2)\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

Using the three equalities: $\sum_{i=1}^{n}(y_i-\bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2 + \sum_{i=1}^{n} e_i^2$ and $R^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 / \sum_{i=1}^{n}(y_i-\bar{y})^2$ and $\hat{\beta}_{yx}^2 \sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2$, we get that the square of the t-ratio (t_{yx}) of the slope coefficient in regression of y on x,

$$\begin{aligned} t_{yx}^2 = \frac{\hat{\beta}_{yx}^2}{s_{yx}^2} &= \frac{n-2}{\sum_{i=1}^{n} e_i^2}\left[\hat{\beta}_{yx}^2 \sum_{i=1}^{n}(x_i-\bar{x})^2\right] \\ &= (n-2)\left[\frac{\sum_{i=1}^{n}(y_i-\bar{y})^2}{\sum_{i=1}^{n} e_i^2}\right]\left[\frac{\hat{\beta}_{yx}^2\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(y_i-\bar{y})^2}\right] \\ &= (n-2)\left[\frac{1}{1-R^2}\right]\left[R^2\right] \end{aligned}$$

Again $R^2 = \rho^2$ implies that square of the t-ratio (t_{yx}) of the slope coefficient in regression of y on x is equal to the square of the t-ratio (t_{xy}) of the slope coefficient in regression of x on y, $t_{yx}^2 = t_{xy}^2$. Also the sign of the slope coefficients in the two regressions are also same as the sign of the covariance between x and y, further implying that $t_{yx} = t_{xy}$.

■

Exercise 4.30 Consider the two regression models

(i) $y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u$

(ii) $y = \gamma_0 + \gamma_1 Z_1 + \gamma_2 Z_2 + v$

where variables Z_1 and Z_2 are distinct from X_1 and X_2. Assume $u \sim \mathcal{N}(0, \sigma_u^2)$ and $v \sim \mathcal{N}(0, \sigma_v^2)$ and the models are estimated using ordinary least squares. If the true model is (i) then which of the following is true?

(a) $\mathbb{E}[\hat{\beta}_1] = \mathbb{E}[\hat{\gamma}_1] = \beta_1$ and $\mathbb{E}[\hat{\sigma}_v^2] = \sigma_u^2$

(b) $\mathbb{E}[\hat{\sigma}_v^2] \geq \sigma_u^2$

(c) $\mathbb{E}[\hat{\sigma}_v^2] \leq \sigma_u^2$

(d) None of the above as the two models cannot be compared

A 4.30

(b) $\mathbb{E}[\hat{\sigma}_v^2] \geq \sigma_u^2$

Let us write the two regressions in Matrix notation: The true model is re-written as $Y = X\beta + u$ where Y is $n \times 1$ vector, X is $n \times 3$ matrix and β is 3×1. Likewise, the false model is $Y = Z\gamma + v$. The estimated coefficient of the regression of Y on Z are:

$$\hat{\gamma} = (Z'Z)^{-1}Z'Y$$

Therefore the residuals of the regression of Y on Z are:

$$Y - Z\hat{\gamma} = Y - Z(Z'Z)^{-1}Z'Y = Y - P_Z Y = (I - P_Z)Y = (I - P_Z)(X\beta + u)$$

So, the residual sum of squares is

$$Y'(I - P_Z)Y = \beta'X'(I - P_Z)X\beta + u'(I - P_Z)u + 2\beta'X'(I - P_Z)u$$

The expected value is therefore

$$\begin{aligned} \mathbb{E}[Y'(I - P_Z)Y] &= \beta'X'(I - P_Z)X\beta + \sigma_u^2[\text{tr}(I - P_Z)] \\ &= \beta'X'(I - P_Z)X\beta + \sigma_u^2(n-3) \end{aligned}$$

Since $\beta'X'(I - P_Z)X\beta \geq 0$ holds, we get the required inequality,

$$\mathbb{E}[\hat{\sigma}_v^2] = \mathbb{E}\left[\frac{[Y'(I - P_Z)Y]}{n-3}\right] = \frac{\mathbb{E}[Y'(I - P_Z)Y]}{n-3} \geq \sigma_u^2$$

The following information is the starting point for Q 31 - 35. Please read it carefully before you proceed to answer.

Consider an economy consisting of N identical firms producing a single final commodity to be used for consumption as well as investment purposes. Each firm is endowed with a Cobb-Douglas production technology, such that

$$Y_t^i = \left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha}; \quad 0 < \alpha < 1$$

where K_t^i and L_t^i denote the amounts of capital and labor employed by the i-th firm at time period t. The final commodity is the numeraire; wage rate for labour (w_t) and the rental rate for capital (r_t) are measured in terms of the final commodity. The firms are perfectly competitive and employ labor and capital so as to maximise their profits - taking the factor prices as given. The aggregate output produced is thus given by:

$$Y_t = \sum_{i=1}^{N}\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha} = (K_t)^{\alpha}(L_t)^{1-\alpha},$$

where $K_t = \sum_{i=1}^{N} K_t^i$ and $L_t = \sum_{i=1}^{N} L_t^i$ are the total capital and labor employed in the aggregate economy in period t.

Labor and Capital on the other hand are provided by the households. There are H identical households, each endowed with k_t^h units of capital and 1 unit of labor at the beginning of period t. Capital stock of the households gets augmented over time due to the savings and investment made by the households. In particular, each household saves and invests exactly half of its total income y_t^h - (which includes its labour as well as capital income) in every period and

consumes the rest, such that $\frac{dk_t^h}{dt} = \frac{1}{2}y_t^h$ (There is no depreciation of capital).
The entire capital endowment at the beginning of every period is supplied inelastically to the market at the given rental rate (r_t). Labor supply however is endogenous and responds to the market wage rate. Out of the total endowment of 1 unit of labour, a household optimally supplies l_t^h units so as to maximise its utility:

$$U_t^h = w_t l_t^h - \left(l_t^h\right)^{\delta}; \quad \delta > 1$$

where the first term captures the (indirect) utility derived from labour earnings while the second term captures the dis-utility of labour.

Exercise 4.31 The labor demand schedule for the aggregate economy is given by the following function:

(a) $L_t = \left[\frac{1}{w_t}\right]^{1/\alpha} K_t$

(b) $L_t = N\left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$

(c) $L_t = \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$

(d) None of the above

A 4.31

(c) $L_t = \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$

To solve for the labor demand, we solve the following optimisation problem:

$$\max_{(K_t, L_t) \geq 0} \quad (K_t)^{\alpha}(L_t)^{1-\alpha} - w_t L_t - r_t K_t$$

Differentiating above with respect to L_t and equating it to 0 gives us labor demand in terms of Capital employed:

$$(1-\alpha)(K_t)^{\alpha}(L_t)^{-\alpha} - w_t = 0$$

$$\Rightarrow \qquad L_t = \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$$

■

Exercise 4.32 The aggregate labour supply schedule by the households is given by the following function:

(a) $L_t^S = \begin{cases} H\left[\frac{w_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \bar{w} \equiv (\delta)^{1/(\delta-1)} \\ H & \text{for } w_t \geq \bar{w} \end{cases}$

(b) $L_t^S = \begin{cases} H\left[\frac{w_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \hat{w} \equiv \delta \\ H & \text{for } w_t \geq \hat{w} \end{cases}$

(c) $L_t^S = \begin{cases} \left[\dfrac{Hw_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \hat{w} \equiv \delta \\ 1 & \text{for } w_t \geq \hat{w} \end{cases}$

(d) None of the above

A 4.32

(b) $L_t^S = \begin{cases} H\left[\dfrac{w_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \hat{w} \equiv \delta \\ H & \text{for } w_t \geq \hat{w} \end{cases}$

To solve for the labor supply, we first solve the following individual household h's optimisation problem:

$$\max_{0 \leq l_t^h \leq 1} \quad w_t l_t^h - \left(l_t^h\right)^\delta$$

Differentiating the objective with respect to l_t^h, we get

$$\frac{dU_t^h}{dl_t^h} = w_t - \delta\left(l_t^h\right)^{\delta-1}$$

Since $\dfrac{dU_t^h}{dl_t^h}$ is monotonically decreasing in l_t^h, therefore the solution will be $l_t^h = 1$ if $\dfrac{dU_t^h}{dl_t^h} \geq 0$ at $l_t^h = 1$ and will satisfy $\dfrac{dU_t^h}{dl_t^h} = 0$ otherwise. Thus, household's labor supply is

$$l_t^h = \begin{cases} \left[\dfrac{w_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \hat{w} \equiv \delta \\ 1 & \text{for } w_t \geq \hat{w} \end{cases}$$

Now there are H identical households, therefore the aggregate supply is

$$L_t^S = \begin{cases} H\left[\dfrac{w_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \hat{w} \equiv \delta \\ H & \text{for } w_t \geq \hat{w} \end{cases}$$

Exercise 4.33 The market clearing wage rate in the short run (period t) is given by:

(a) $w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < \dfrac{H}{\delta} \equiv \hat{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \hat{K} \end{cases}$

(b) $w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < H\left(\dfrac{\delta}{1-\alpha}\right)^{1/\alpha} \equiv \bar{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$

(c) $w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < H\left(\dfrac{\delta}{1-\alpha}\right)^{1/\alpha} \equiv \bar{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$

(d) None of the above

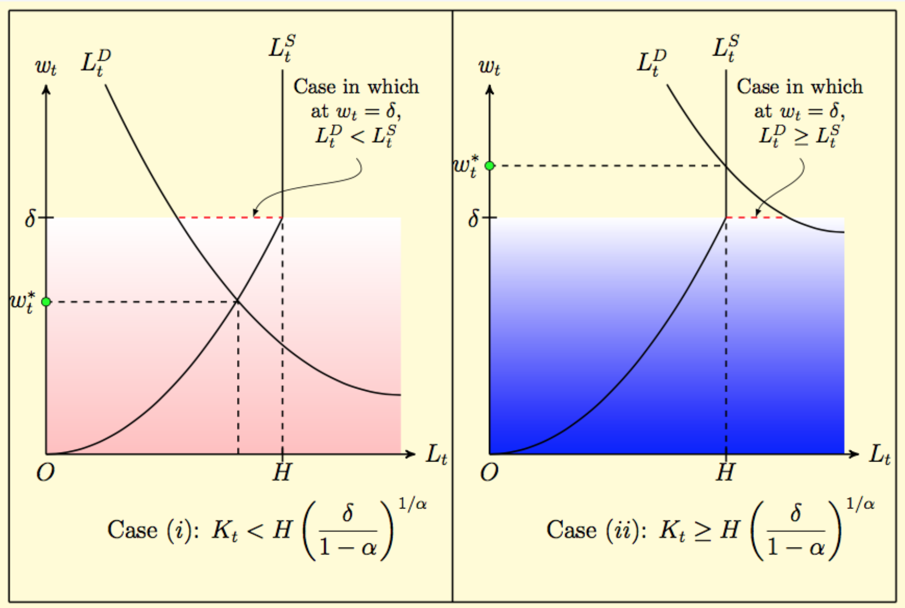
A 4.33

(c) $w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < H\left(\dfrac{\delta}{1-\alpha}\right)^{1/\alpha} \equiv \bar{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$

Solving for household's utility maximisation problem and firm's profit maximisation problem, we get the following as aggregate labor supply and demand respectively,

$$L_t^S = \begin{cases} H\left[\dfrac{w_t}{\delta}\right]^{1/(\delta-1)} & \text{for } w_t < \hat{w} \equiv \delta \\ H & \text{for } w_t \geq \hat{w} \end{cases}$$

$$L_t^D = \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$$



Solving the system for the equilibrium wage we get,

$$w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < H\left(\dfrac{\delta}{1-\alpha}\right)^{1/\alpha} \equiv \bar{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$$

Exercise 4.34 Equilibrium output in the short run (period t):

(a) is a strictly convex function of K_t for $K_t < H\left(\frac{\delta}{1-\alpha}\right)^{1/\alpha} \equiv \bar{K}$; and is a strictly concave function of K_t for $K_t \geq \bar{K}$
(b) is a strictly concave function of K_t for all values of K_t
(c) is a strictly convex function of K_t for all values of K_t
(d) is a linear function of K_t for all values of K_t

A 4.34

(b) is a strictly concave function of K_t for all values of K_t

Proceeding from where we left in Q 33, we have got the equilibrium wage as

$$w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < H\left(\dfrac{\delta}{1-\alpha}\right)^{1/\alpha} \equiv \bar{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$$

The corresponding level of equilibrium employment is

$$L_t^* = \begin{cases} H\left[\dfrac{1}{\delta}\right]^{\frac{1}{\delta-1}}\left[\dfrac{K_t(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha}{\alpha+\delta-1}} & \text{for } K_t < \bar{K} \\ H & \text{for } K_t \geq \bar{K} \end{cases}$$

Therefore, the equilibrium output is

$$Y_t^* = \begin{cases} H^{1-\alpha}\left[\dfrac{1}{\delta}\right]^{\frac{1-\alpha}{\delta-1}}\left[\dfrac{K_t(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(1-\alpha)}{\alpha+\delta-1}} K_t^{\alpha} & \text{for } K_t < \bar{K} \\ H^{1-\alpha}K_t^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$$

We can take out K_t from the bracket and rewrite Y_t^* as

$$Y_t^* = \begin{cases} H^{1-\alpha}\left[\dfrac{1}{\delta}\right]^{\frac{1-\alpha}{\delta-1}}\left[\dfrac{(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(1-\alpha)}{\alpha+\delta-1}} K_t^{\alpha\delta/(\alpha+\delta-1)} & \text{for } K_t < \bar{K} \\ H^{1-\alpha}K_t^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$$

Since, $0 < \alpha < \alpha\delta/(\alpha+\delta-1) < 1$ and Y_t^* is continuous, therefore Y_t^* is a strictly concave function of K_t for all values of K_t (i.e. when $K_t < \bar{K}$ and $K_t \geq \bar{K}$).

■

Exercise 4.35 Over time the aggregate output in this economy

(a) initially increases until $K_t < \bar{K}$, and then reaches a constant value within finite time when $K_t \geq \bar{K}$
(b) initially decreases until $K_t < \bar{K}$, and then reaches a constant value within finite time when $K_t \geq \bar{K}$
(c) keep increasing at a decreasing rate and approaches a constant value only in the very long run (when $t \to \infty$)
(d) increases at a constant rate until $K_t < \bar{K}$; increases at a decreasing rate when $K_t \geq \bar{K}$ and

approaches a constant value only in the very long run (when $t \to \infty$)

A 4.35

(c) keep increasing at a decreasing rate and approaches a constant value only in the very long run (when $t \to \infty$)

$$Y_t^* = \begin{cases} H^{1-\alpha}\left[\frac{1}{\delta}\right]^{\frac{1-\alpha}{\delta-1}}\left[\frac{(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(1-\alpha)}{\alpha+\delta-1}} K_t^{\alpha\delta/(\alpha+\delta-1)} & \text{for } K_t < \bar{K} \\ H^{1-\alpha}K_t^{\alpha} & \text{for } K_t \geq \bar{K} \end{cases}$$

Define $A \equiv H^{1-\alpha}\left[\frac{1}{\delta}\right]^{\frac{1-\alpha}{\delta-1}}\left[\frac{(1-\alpha)^{1/\alpha}(\delta)^{1/(\delta-1)}}{H}\right]^{\frac{\alpha(1-\alpha)}{\alpha+\delta-1}}$. Applying log to Y_t^* gives us

$$\log Y_t^* = \begin{cases} \log A + \frac{\alpha\delta}{(\alpha+\delta-1)}\log K_t & \text{for } K_t < \bar{K} \\ (1-\alpha)\log H + \alpha \log K_t & \text{for } K_t \geq \bar{K} \end{cases}$$

We will differentiate $\log Y_t^*$ with respect to t and apply chain rule: $\frac{d\log Y_t^*}{dt} = \frac{d\log Y_t^*}{dK_t} \times \frac{dK_t}{dt}$.

Using $K_t = Hk_t^h, Y_t = Hy_t^h$, and $\frac{dk_t^h}{dt} = \frac{1}{2}y_t^h$, we get $\frac{dK_t}{dt} = \frac{1}{2}Y_t$. Therefore,

$$\frac{d\log Y_t^*}{dt} = \begin{cases} \frac{\alpha\delta}{(\alpha+\delta-1)K_t} \times \frac{Y_t^*}{2} & \text{for } K_t < \bar{K} \\ \frac{\alpha}{K_t} \times \frac{Y_t^*}{2} & \text{for } K_t > \bar{K} \end{cases}$$

Since Y_t^* is concave and K_t increases with time, $\left(\frac{dK_t}{dt} = \frac{1}{2}Y_t > 0\right)$. Therefore, $\frac{Y_t^*}{K_t}$ falls as K_t^* rises causing the rate of growth of output to fall with time. $\frac{Y_t^*}{K_t} = \frac{H^{1-\alpha}}{K_t^{1-\alpha}}$ for $K_t > \bar{K}$ implies that the rate of growth of output converges to zero in the very long run (as $t \to \infty$).

The following information is the starting point for Q 36 - 40. Please read it carefully before you proceed to answer.
Consider an economy consisting of N identical firms producing a single final commodity to be used for consumption as well as investment purposes. Each firm is endowed with a Cobb-Douglas production technology, such that

$$Y_t^i = \left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{1-\alpha}; \quad 0 < \alpha < 1$$

where K_t^i and L_t^i denote the amounts of capital and labor employed by the i-th firm at time period t. The final commodity is the numeraire; wage rate for labour (w_t) and the rental rate for capital (r_t) are measured in terms of the final commodity. The firms are perfectly competitive and employ labor and capital so as to maximise their profits - taking the factor prices as given.

The aggregate output produced is thus given by:

$$Y_t = \sum_{i=1}^{N} \left(K_t^i\right)^{\alpha} \left(L_t^i\right)^{1-\alpha} = \left(K_t\right)^{\alpha} \left(L_t\right)^{1-\alpha},$$

where $K_t = \sum_{i=1}^{N} K_t^i$ and $L_t = \sum_{i=1}^{N} L_t^i$ are the total capital and labor employed in the aggregate economy in period t.
Labor and Capital on the other hand are provided by the households. There are H identical households, each endowed with k_t^h units of capital and 1 unit of labor at the beginning of period t. Capital stock of the households gets augmented over time due to the savings and investment made by the households. In particular, each household saves and invests exactly half of its total income y_t^h - (which includes its labour as well as capital income) in every period and consumes the rest, such that $\frac{dk_t^h}{dt} = \frac{1}{2} y_t^h$ (There is no depreciation of capital).
The entire capital endowment at the beginning of every period is supplied inelastically to the market at the given rental rate (r_t). Labor supply however is endogenous and responds to the market wage rate. Out of the total endowment of 1 unit of labour, a household optimally supplies l_t^h units so as to maximise its utility:

$$U_t^h = \begin{cases} w_t l_t^h - D \; (D > 0) & \text{for } l_t^h > 0 \\ 0 & \text{for } l_t^h = 0 \end{cases}$$

For the case $l_t^h > 0$, the first term captures the (indirect) utility derived from labour earnings while the constant term D captures the dis-utility of labour - which is independent of quantity of labour supplied.

Exercise 4.36 The new aggregate labour supply schedule by the households is given by the following function:

(a) $L_t^S = \begin{cases} 0 & \text{for } w_t < \underline{w} \equiv \dfrac{D}{H} \\ H & \text{for } w_t \geq \underline{w} \end{cases}$

(b) $L_t^S = H$ for all values of w_t

(c) $L_t^S = \begin{cases} 0 & \text{for } w_t < \tilde{w} \equiv D \\ H & \text{for } w_t \geq \tilde{w} \end{cases}$

(d) None of the above

A 4.36

(c) $L_t^S = \begin{cases} 0 & \text{for } w_t < \tilde{w} \equiv D \\ H & \text{for } w_t \geq \tilde{w} \end{cases}$

To solve for the labor supply, we first solve the following individual household h's optimisation problem:

$$\max_{0 \leq l_t^h \leq 1} \quad U_t^h$$

Differentiating the objective with respect to l_t^h, we get

$$\frac{dU_t^h}{dl_t^h} = w_t > 0 \quad \text{for } l_t^h > 0$$

Since U_t^h is discontinuous at $l_t^h = 0$ and is an increasing function for $l_t^h > 0$, therefore the solution will be either $l_t^h = 1$ or $l_t^h = 0$ depending on the utility of the household. Thus, household's labor supply is

$$l_t^h = \begin{cases} 0 & \text{for } w_t < \tilde{w} \equiv D \\ 1 & \text{for } w_t \geq \tilde{w} \end{cases}$$

Now there are H identical households, therefore the aggregate supply is

$$L_t^S = \begin{cases} 0 & \text{for } w_t < \tilde{w} \equiv D \\ H & \text{for } w_t \geq \tilde{w} \end{cases}$$

■

Exercise 4.37 The new market clearing wage rate in the short run (period t) is given by:

(a) $w_t^* = \begin{cases} D & \text{for } K_t < H\left(\dfrac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \tilde{K} \end{cases}$

(b) $w_t^* = D$ for all values of K_t

(c) $w_t^* = \begin{cases} \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\frac{\alpha(\delta-1)}{\alpha+\delta-1}} & \text{for } K_t < \dfrac{H}{D} \equiv \hat{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \hat{K} \end{cases}$

(d) None of the above

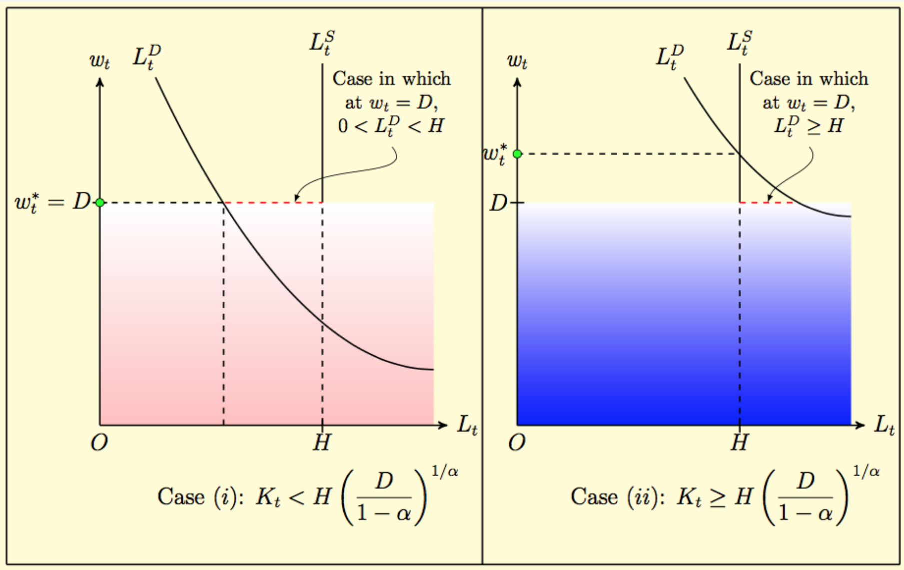
A 4.37

(a) $w_t^* = \begin{cases} D & \text{for } K_t < H\left(\dfrac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \tilde{K} \end{cases}$

Solving for household's utility maximisation problem and firm's profit maximisation problem, we get the following as aggregate labor supply and demand respectively,

$$L_t^S = \begin{cases} 0 & \text{for } w_t < \tilde{w} \equiv D \\ H & \text{for } w_t \geq \tilde{w} \end{cases}$$

$$L_t^D = \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$$



Solving the system for the equilibrium wage we get,

$$w_t^* = \begin{cases} D & \text{for } K_t < H\left(\dfrac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ \left[\dfrac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \tilde{K} \end{cases}$$

Exercise 4.38 An increase in the number of firms (N)

(a) leaves the wage rate unchanged in the short run (until $K_t < \tilde{K}$) and increases it thereafter
(b) increases the wage rate in the short run (until $K_t < \hat{K}$) and leaves it unchanged thereafter
(c) leaves the wage rate unchanged irrespective of K_t
(d) None of the above

A 4.38

(c) leaves the wage rate unchanged irrespective K_t

Both the demand and the supply curves for labor are independent of the number of firms, N:

$$L_t^S = \begin{cases} 0 & \text{for } w_t < \tilde{w} \equiv D \\ H & \text{for } w_t \geq \tilde{w} \end{cases}$$

$$L_t^D = \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha} K_t$$

Therefore, an increase in the number of firms leaves the wage rate unchanged irrespective of K_t.

Exercise 4.39 The new equilibrium output in the short run (period t)

(a) is a linear function of K_t for $K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K}$, and is strictly concave function of K_t for $K_t \geq \tilde{K}$
(b) is a strictly concave function of K_t for all values of K_t
(c) is a strictly convex function of K_t for all values of K_t
(d) is a linear function of K_t for all values of K_t

A 4.39

(a) is a linear function of K_t for $K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K}$, and is strictly concave function of K_t for $K_t \geq \tilde{K}$

We have already solved for the equilibrium wage in Q 37 above:

$$w_t^* = \begin{cases} D & \text{for } K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ \left[\frac{K_t(1-\alpha)^{1/\alpha}}{H}\right]^{\alpha} & \text{for } K_t \geq \tilde{K} \end{cases}$$

The corresponding equilibrium level of employment is

$$L_t^* = \begin{cases} \left[\frac{1-\alpha}{D}\right]^{1/\alpha} K_t & \text{for } K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ H & \text{for } K_t \geq \tilde{K} \end{cases}$$

Therefore, the equilibrium output is

$$Y_t^* = \begin{cases} \left[\frac{1-\alpha}{D}\right]^{(1-\alpha)/\alpha} K_t & \text{for } K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ H^{1-\alpha} K_t^{\alpha} & \text{for } K_t \geq \tilde{K} \end{cases}$$

Clearly, Y_t^* s a linear function of K_t for $K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K}$, and is strictly concave function of K_t for $K_t \geq \tilde{K}$.

Exercise 4.40 Over time the aggregate output in this economy

(a) initially increases until $K_t < \tilde{K}$, and then reaches a constant value within finite time when $K_t \geq \tilde{K}$
(b) initially decreases until $K_t < \tilde{K}$, and then reaches a constant value within finite time when $K_t \geq \tilde{K}$
(c) increases at a constant rate until $K_t < \tilde{K}$; increases at a decreasing rate when $K_t \geq \tilde{K}$ and approaches a constant value only in the very long run (when $t \to \infty$)
(d) None of the above

A 4.40

(c) increases at a constant rate until $K_t < \tilde{K}$; increases at a decreasing rate when $K_t \geq \tilde{K}$ and approaches a constant value only in the very long run (when $t \to \infty$)

From the previous problem we get,

$$Y_t^* = \begin{cases} \left[\frac{1-\alpha}{D}\right]^{(1-\alpha)/\alpha} K_t & \text{for } K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ H^{1-\alpha}K_t^{\alpha} & \text{for } K_t \geq \tilde{K} \end{cases}$$

Again using $\frac{dK_t}{dt} = \frac{1}{2}Y_t$ and $\frac{d\log Y_t^*}{dt} = \frac{d\log Y_t^*}{dK_t} \times \frac{dK_t}{dt}$, we get

$$\frac{d\log Y_t^*}{dt} = \begin{cases} \frac{Y_t^*}{2K_t} & \text{for } K_t < H\left(\frac{D}{1-\alpha}\right)^{1/\alpha} \equiv \tilde{K} \\ \frac{\alpha Y_t^*}{2K_t} & \text{for } K_t > \tilde{K} \end{cases}$$

$\frac{Y_t^*}{K_t}$ is constant for $K_t < \bar{K}$ and Y_t^* is strictly concave for $K_t > \bar{K}$ causing fall in value of $\frac{\alpha Y_t^*}{K_t}$ as K_t increases. We know that K_t^* increases with time and rate of growth of Y_t^* only depends on $\frac{Y_t^*}{K_t}$. Therefore, Y_t^* increases at a constant rate until $K_t < \tilde{K}$; increases at a decreasing rate when $K_t \geq \tilde{K}$ and approaches a constant value only in the very long run (when $t \to \infty$).

Exercise 4.41 $\lim_{n\to\infty} \sqrt[n]{n} =$

(a) 0
(b) 0.5
(c) 1
(d) 2

A 4.41

(c) 1

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{n} &= \lim_{n\to\infty} e^{\ln(\sqrt[n]{n})} \\
&= e^{\lim_{n\to\infty} \ln(\sqrt[n]{n})} \\
&= e^{\lim_{n\to\infty} \frac{1}{n}\ln(n)} \\
&= e^{\lim_{n\to\infty} \frac{1}{n^2}} \ldots \text{(By L'Hôpital's rule)} \\
&= e^0 \\
&= 1
\end{aligned}$$

Exercise 4.42 $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) =$

(a) -1
(b) 0
(c) 1
(d) The limit does not exist

A 4.42

(b) 0

$$\begin{array}{rcl}
& -1 \leq \cos\left(\frac{1}{x}\right) \leq 1 & \forall x \in \mathbb{R} \\
\Rightarrow & -x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2 & \forall x \in \mathbb{R} \\
\Rightarrow & \lim_{x\to 0} -x^2 \leq \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x^2 & \\
\Rightarrow & 0 \leq \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) \leq 0 & \\
\Rightarrow & \lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0 &
\end{array}$$

Exercise 4.43 Suppose $A_1, A_2, \ldots$ is a countably infinite family of subsets of a vector space. Suppose all of these sets are linearly independent, and that $A_1 \subseteq A_2 \subseteq \cdots$ Then $\cup_{i=1}^{\infty} A_i$ is

(a) a linearly independent set of vectors
(b) a linearly dependent set of vectors
(c) linearly independent provided the vectors are orthogonal
(d) not necessarily either dependent or independent

A 4.43

(a) a linearly independent set of vectors

A subset S of a vector space V is called *linearly dependent* if there exist a finite number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

For any vectors $v_1, v_2, \ldots, v_n$ we have that

$$0v_1 + 0v_2 + \cdots + 0v_n = 0$$

This is called the *trivial representation* of 0 as a linear combination of $v_1, v_2, \ldots, v_n$. A subset S of a vector space V is then said to be *linearly independent* if it is not linearly dependent, in other words, a set is linearly independent if the only representations of 0 as a linear combination of its vectors are trivial representations. If $A_1, A_2, \ldots$ is a countably infinite family of linearly independent subsets of a vector space such that $A_1 \subseteq A_2 \subseteq \cdots$. We will show that $\cup_{i=1}^{\infty} A_i$ is also linearly independent. Consider an arbitrary finite collection of distinct vectors $v_1, v_2, \ldots, v_n$ from set $\cup_{i=1}^{\infty} A_i$. Since $v_1, v_2, \ldots, v_n \in \cup_{i=1}^{\infty} A_i$ there exist $A_{k_1}, A_{k_2}, \ldots, A_{k_n}$ such that $v_i \in A_{k_i}, \forall\ 1 \leq i \leq n$. Given that the collection (A_j) satisfy $A_1 \subseteq A_2 \subseteq \cdots$, this implies that for $p = \max\{k_1, k_2, \ldots, k_n\}$, we have $v_i \in A_p, \forall\ 1 \leq i \leq n$. Since A_p is independent, the only set of scalars $a_1, a_2, \ldots, a_n$, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

is $a_1 = a_2 = \cdots = a_n = 0$. Therefore, $\cup_{i=1}^{\infty} A_i$ is independent.

■

Exercise 4.44 If u and v are distinct vectors and k and t are distinct scalars, then the vectors $u + k(u - v)$ and $u + t(u - v)$

(a) are linearly independent
(b) may be identical
(c) are linearly dependent
(d) are distinct

A 4.44

(d) are distinct

We will first rule out options (a) and (c). Consider $u = (1,0)$, $v = (0,1)$, $k = 0$, $t = -1$. For this example, $u + k(u - v) = (1,0)$ and $u + t(u - v) = (0,1)$ are independent. This rules out (c). Consider $u = (0,0)$, $v = (1,1)$, $k = 0$, $t = -1$. For this example, $u + k(u - v) = (0,0)$ and $u + t(u - v) = (1,1)$ are dependent. This rules out (a). We will now show that $u + k(u - v)$ and $u + t(u - v)$ are always distinct. Since u and v are distinct vectors and k and t are distinct scalars, this implies $(u - v) \neq 0$ which further implies that $k(u - v) \neq t(u - v)$. Hence, we get $u + k(u - v) \neq u + t(u - v)$.

■

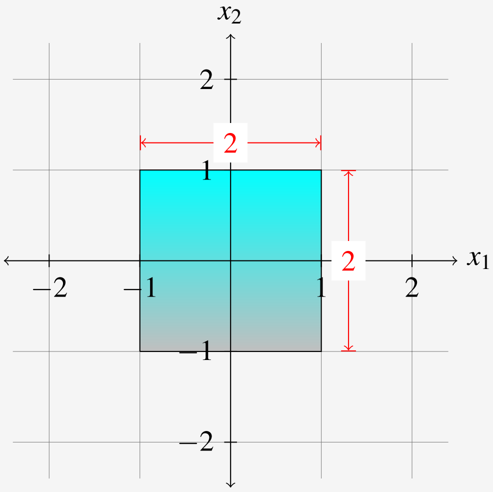
Exercise 4.45 Let $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ be the distance between two points (x_1, x_2) and (y_1, y_2) on the plane. Then the locus of points at distance 1 from the origin is

(a) a square with side length $= 1$
(b) a square with side length $= \sqrt{2}$

(c) a square with side length $= 2$
(d) a circle with radius $= 1$

A 4.45

(c) a square with side length $= 2$



The locus of points (x_1, x_2) at distance 1 from the origin satisfy

$$d((x_1,x_2),(0,0))$$

$$= \max\{|x_1|,|x_2|\}$$

$$= 1.$$

Equivalently, these are the points (x_1, x_2) satisfying

$$(|x_1| = 1, -1 \le x_2 \le 1) \quad \text{or}$$
$$(|x_2| = 1, -1 \le x_1 \le 1)$$

These points are represented in the graph as the boundary of the shaded square with side length $= 2$.

Exercise 4.46 The set of all pairs of positive integers a, b with $a < b$ such that $a^b = b^a$
(a) is an empty set
(b) consists of a single pair
(c) consists of multiple, but finite number of pairs
(d) is countably infinite

A 4.46

(b) consists of a single pair

Finding the set of all pairs of positive integers a, b with $a < b$ such that $a^b = b^a$ is equivalent to finding set of all such pairs with $b \ln a = a \ln b$ or

$$\frac{\ln a}{a} = \frac{\ln b}{b}$$

In order to solve the above, we will first find out how the following function behaves

$$f(x) \equiv \frac{\ln x}{x}$$

Taking its derivative, we get

$$f'(x) = \frac{1 - \ln x}{x^2}$$

Therefore, f is a strictly increasing function in the interval $(0, e)$ and it is a strictly decreasing function in the interval (e, ∞). Since we are looking for positive integers a, b with $a < b$ such that $f(a) = f(b)$, a cannot be greater than e because $e < a < b$ implies $f(a) > f(b)$. Thus, $a < e$. So, we just need to check for two values: $a = 1$ and $a = 2$. Consider $a = 1$ first. This gives us $f(a) = f(1) = 0$ and we know for all values of $b > 1$, it is the case that $f(b) > 0$. Therefore, there is no solution for $a = 1$. Next we consider $a = 2$. Given that f is a strictly increasing function in the interval $(0, e)$ and it is a strictly decreasing function in the interval (e, ∞), there is at most one value of b that will solve $f(b) = f(a)$. This value of b is 4. Hence, the set of all pairs of positive integers a, b with $a < b$ such that $a^b = b^a$ consists of a single pair $(a, b) = (2, 4)$.

Exercise 4.47 Suppose c is a given positive real number. The equation $\ln x = cx^2$ must have a solution if

(a) $c < 1/(2e)$
(b) $c < 1/e$
(c) $c > 1/(2e)$
(d) $c > (1/e)$

A 4.47

(a) $c < 1/(2e)$

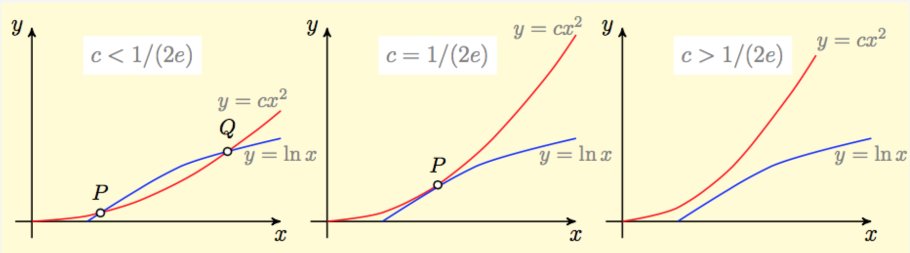
Given that $y = cx^2$ is a strictly convex function and $y = \ln x$ is a strictly concave function, they can intersect each other at most two times. We will find the value of c for which the intersection happens exactly once. This will give us a cutoff value of c above which the intersection will never happen and below which the functions intersect exactly twice. Note that if $y = cx^2$ and $y = \ln x$ intersect exactly once then they must be tangential to each other at that point. Therefore, at the point of tangency, it must be the case that the functions take the same value

$$\ln x = cx^2$$

and have the same slope

$$1/x = 2cx$$

Solving the two equations for c and x, we get the point of tangency as $x = e^{1/2}$ and the value of c for which the tangency occurs is $c = 1/(2e)$. Clearly, $c \leq 1/(2e)$ implies that $\ln x = cx^2$ must have a solution. Therefore, $c < 1/(2e)$ also implies that $\ln x = cx^2$ must have a solution.



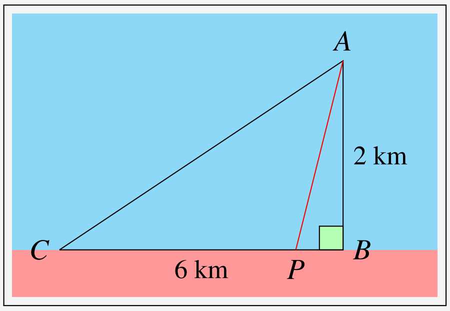
■

Exercise 4.48 Sania's boat is at point A on the sea. The closest point on land, point B, is 2 km away. Point C on land is 6 km from point B, such that triangle (ABC) is right-angled at point B. Sania wishes to reach point C, by rowing to some point P on the line $\bar{BC}$, and jog the remaining distance to C. If she rows 2 km per hour and jogs 5 km per hour, at what distance from point B should she choose her landing point P, in order to minimise her time to reach point C?

(a) $21/\sqrt{4}$
(b) $4/\sqrt{21}$
(c) $4/\sqrt{12}$
(d) $21/\sqrt{21}$

A 4.48

(b) $4/\sqrt{21}$



Sania wants to choose P to minimise her total travel time from A to C, that is, the rowing time from A to P *plus* the jogging time from P to C. Let us denote the distance between B and P by x. Therefore, the problem is to minimise the following with respect to x:

$$\frac{\overline{AP}}{2}+\frac{\overline{PC}}{5}=\frac{\sqrt{2^2+x^2}}{2}+\frac{6-x}{5}$$

Solving the first order condition we get $x=4/\sqrt{21}$.

Exercise 4.49 Suppose $A_j, j=1,2,\ldots$ are non-empty sets of real numbers. Define the sets $C_n=\cap_{k=n}^{\infty}\cup_{j=k}^{\infty}A_j, n=1,2,\ldots$. Which of the choices below must then hold for a given n? (where the symbol $\subset$ stands for 'strict subset').

(a) $C_n\subset C_{n+1}$
(b) $C_{n+1}\subset C_n$
(c) $C_n=C_{n+1}$
(d) None of the above need hold

A 4.49

(c) $C_n=C_{n+1}$

Define $B_k\equiv\cup_{j=k}^{\infty}A_j$. Note that $B_1\supseteq B_2\supseteq\cdots B_n\supseteq B_{n+1}\cdots$.

$$\begin{aligned}C_n &= \cap_{k=n}^{\infty}B_k\\ &= \left[\cap_{k=n+1}^{\infty}B_k\right]\cap B_n\end{aligned}$$

Since $\cap_{k=n+1}^{\infty}B_k\subseteq B_{n+1}\subseteq B_n$, we get

$$\begin{aligned}C_n &= \left[\cap_{k=n+1}^{\infty}B_k\right]\\ &= C_{n+1}\end{aligned}$$

Exercise 4.50 Suppose x and y are given integers. Consider the following statements:

A. If $2x+3y$ is divisible by 17, then $9x+5y$ is divisible by 17.
B. If $9x+5y$ is divisible by 17, then $2x+3y$ is divisible by 17.

Which of the following is true?

(a) A is true and B is false
(b) B is true and A is false
(c) Both A and B are true
(d) Neither A nor B is true

A 4.50

(c) Both A and B are true

Let us denote "a divides b" by $a \mid b$ in what follows:

$$\begin{aligned} & 17 \mid 2x+3y \\ \Leftrightarrow \quad & 17 \mid -8x-12y \\ \Leftrightarrow \quad & 17 \mid 17x+17y+(-8x-12y) \\ \Leftrightarrow \quad & 17 \mid 9x+5y \end{aligned}$$

The following information is the starting point for Q 51 - 52. Consider an exchange economy with two goods. Suppose agent i and j have the same preferences. Moreover, suppose their preferences have the following property: if (a,b) and (c,d) are distinct bundles that are indifferent to each other, then the bundle $((a+c)/2,(b+d)/2)$ is strictly preferred to (a,b) and (c,d).

Exercise 4.51 In a Pareto efficient allocation, i and j

(a) will get the same bundle
(b) may get different bundles
(c) will get the same bundle, provided their endowments are identical
(d) will get the same bundle, provided their endowments are identical and the preferences are monotonically increasing

A 4.51

(b) may get different bundles

The following example of an exchange economy justifies the correct option:
Consider an exchange economy with two agents, A and B, and two goods, X and Y. Agent A's endowment is $(1,1)$ and agent B's endowment is $(1,1)$. Preferences of agents A and B are as follows,

$$u_A(x_A,y_A)=\sqrt{x_A}+\sqrt{y_A} \quad u_B(x_B,y_B)=\sqrt{x_B}+\sqrt{y_B},$$

where A consumes x_A and y_A units of X and Y respectively, similarly B's consumption is x_B and y_B. In this economy a feasible allocation $((x_A,y_A),(x_B,y_B))=((0,0),(2,2))$ is Pareto optimal but the consumers get different bundles.

Exercise 4.52 In a competitive equilibrium allocation, i and j

(a) will get the same bundle
(b) may get different bundles
(c) will get the same bundle, if their endowments are identical
(d) will get the same bundle, only if their endowments are identical and the preferences are monotonically increasing

A 4.52

(c) will get the same bundle, if their endowments are identical

To prove (c), consider an exchange economy with two agents, A and B, and two goods, X and Y. Agent A's endowment is (α,β) and agent B's endowment is also (α,β). Preferences of agents A and B are identical and have the property: if (a,b) and (c,d) are distinct bundles that are indifferent to each other, then the bundle $((a+c)/2,(b+d)/2)$ is strictly preferred to (a,b) and (c,d), in short, averages are better than extremes. Suppose $((x_A^*,y_A^*),(x_B^*,y_B^*))$ is the competitive equilibrium allocation and (p_x^*,p_y^*) is the supporting price vector. Since both the individuals have identical preferences, identical endowments and face the same set of prices, they also have identical budgets. Therefore, (x_A^*,y_A^*) and (x_B^*,y_B^*) are both utility maximising bundles given the budget constraint $p_x^*x+p_y^*y \leq p_x^*\alpha+p_y^*\beta$. Thus, both individuals are indifferent between choosing (x_A^*,y_A^*) and (x_B^*,y_B^*) given the prices. If (x_A^*,y_A^*) and (x_B^*,y_B^*) are distinct, then given that the budget set is convex, and the preference satisfy the property that averages are better than extremes, this implies that the bundle $((x_A^*+x_B^*)/2,(y_A^*+y_B^*)/2)$ is strictly preferred and affordable, contradicting that (x_A^*,y_A^*) and (x_B^*,y_B^*) are the utility maximising choice. Thus, (x_A^*,y_A^*) and (x_B^*,y_B^*) must be equal.

The following information is the starting point for Q 53 - 55. Two firms produce the same commodity. Let x_1 and x_2 be the quantity choices of firms 1 and 2 respectively. The total quantity is $X = x_1+x_2$. The inverse demand function is $P = a-bX$, where P is the market price, and a and b are the intercept and the slope parameters respectively. Firms 1 and 2 have constant average costs equal to c_1 and c_2 respectively. Suppose $b > 0$, $0 < c_1 < c_2 < a$ and $a+c_1 > 2c_2$.

Exercise 4.53 In a Cournot equilibrium,

(a) firm 1 has the larger market share and the larger profit
(b) firm 2 has the larger market share and the larger profit
(c) firm 1 has the larger market share and the smaller profit
(d) firm 2 has the larger market share and the smaller profit

A 4.53

(a) firm 1 has the larger market share and the larger profit

Situation described here is a Cournot Duopoly Game with asymmetric costs. We will solve this problem by first finding the best response of firm $i \in \{1,2\}$. Firm i solves the following problem:

$$\max_{x_i \geq 0} \quad (a - b(x_i + x_j))x_i - c_i x_i, \quad j \neq i$$

Differentiating the objective for both the firms and solving the first order condition we get

$$x_1 = \frac{a - c_1 - bx_2}{2b}, \quad x_2 = \frac{a - c_2 - bx_1}{2b}$$

Solving the system, we obtain

$$x_1 = \frac{a - 2c_1 + c_2}{3b}, \quad x_2 = \frac{a - 2c_2 + c_1}{3b}$$

And the respective market shares are

$$s_1 = \frac{a - 2c_1 + c_2}{2a - c_1 - c_2}, \quad s_2 = \frac{a - 2c_2 + c_1}{2a - c_1 - c_2}$$

Profits in equilibrium is given by

$$\pi_1 = \frac{(a - 2c_1 + c_2)^2}{9b}, \quad \pi_2 = \frac{(a - 2c_2 + c_1)^2}{9b}$$

Since $c_1 < c_2$, we get $s_1 > s_2$ and $\pi_1 > \pi_2$.

■

Exercise 4.54 If a increases, then

(a) the market share of firm 1 increases and price increases
(b) the market share of firm 1 decreases and price increases
(c) the market share of firm 1 increases and price decreases
(d) the market share of firm 1 decreases and price decreases

A 4.54

(b) the market share of firm 1 decreases and price increases

Situation described here is a Cournot Duopoly Game with asymmetric costs. We will solve this problem by first finding the best response of firm $i \in \{1,2\}$. Firm i solves the following problem:

$$\max_{x_i \geq 0} \quad (a - b(x_i + x_j))x_i - c_i x_i, \quad j \neq i$$

Differentiating the objective for both the firms and solving the first order condition we get

$$x_1 = \frac{a - c_1 - bx_2}{2b}, \quad x_2 = \frac{a - c_2 - bx_1}{2b}$$

Solving the system, we obtain

$$x_1 = \frac{a - 2c_1 + c_2}{3b}, \quad x_2 = \frac{a - 2c_2 + c_1}{3b}$$

And the respective market shares are

$$s_1 = \frac{a - 2c_1 + c_2}{2a - c_1 - c_2}, \quad s_2 = \frac{a - 2c_2 + c_1}{2a - c_1 - c_2}$$

Market price is given by

$$p = a - b(x_1 + x_2) = \frac{a + c_1 + c_2}{3}$$

Differentiating s_1 and p with respect to a, we get

$$\frac{ds_1}{da} = \frac{3(c_1 - c_2)}{(2a - c_1 - c_2)^2} < 0 \quad \frac{dp}{da} = \frac{1}{3} > 0$$

Therefore, if a increases, then the market share of firm 1 decreases and price increases.

Exercise 4.55 If b decreases, then

(a) the price and market share of firm 1 increase
(b) the price and market share of firm 1 decrease
(c) the market shares are unchanged and price increases
(d) neither price, nor market shares, change

A 4.55

(d) neither price, nor market shares, change

Situation described here is a Cournot Duopoly Game with asymmetric costs. We will solve this problem by first finding the best response of firm $i \in \{1,2\}$. Firm i solves the following problem:

$$\max_{x_i \geq 0} \quad (a - b(x_i + x_j))x_i - c_i x_i, \quad j \neq i$$

Differentiating the objective for both the firms and solving the first order condition we get

$$x_1 = \frac{a - c_1 - bx_2}{2b}, \quad x_2 = \frac{a - c_2 - bx_1}{2b}$$

Solving the system, we obtain

$$x_1 = \frac{a - 2c_1 + c_2}{3b}, \quad x_2 = \frac{a - 2c_2 + c_1}{3b}$$

And the respective market shares are

$$s_1 = \frac{a - 2c_1 + c_2}{2a - c_1 - c_2}, \quad s_2 = \frac{a - 2c_2 + c_1}{2a - c_1 - c_2}$$

Market price is given by

$$p = a - b(x_1 + x_2) = \frac{a + c_1 + c_2}{3}$$

Neither the market share, nor the price is a function of b, therefore if b decreases, then neither price, nor market shares change.

■

Exercise 4.56 Suppose that an economy has endowment of K units of capital and L units of labour. Two final goods X_1 and X_2 can be produced by the following technologies,

$$X_1 = \sqrt{kl}, \quad X_2 = \sqrt{l}$$

where k is quantity of capital and l is quantity of labour. Find the production possibility frontier.

(a) $X_1^2 + KX_2^2 = KL$
(b) $X_1^2 + X_2^2 = KL$
(c) $X_1 + \sqrt{K}X_2 = \sqrt{KL}$
(d) $X_1 + X_2^2 = KL$

A 4.56

(a) $X_1^2 + KX_2^2 = KL$

Any efficient allocation $((l_1, k_1), (l_2, k_2))$ of inputs between the production of two outputs X_1 and X_2 must satisfy $k_2 = 0$. This gives us

$$
\begin{aligned}
X_1 &= \sqrt{Kl_1} \\
X_2 &= \sqrt{l_2} \\
L &= l_1 + l_2
\end{aligned}
$$

Any combination (X_1, X_2) satisfying the above set of conditions is efficient. Thus, production possibility frontier can be obtained by eliminating l_1 and l_2 from the system. The first two from the above set of equations can be re-written as

$$
\begin{aligned}
l_1 &= \frac{X_1^2}{K} \\
l_2 &= X_2^2
\end{aligned}
$$

Substituting them in the third equation we get

$$\frac{X_1^2}{K} + X_2^2 = L$$

Alternatively,

$$X_1^2 + KX_2^2 = KL$$

■

Exercise 4.57 A two-person two commodity economy has social endowment of $x = 1$ unit of food and $y = 1$ unit of wine. Agents preferences are increasing in own consumption but decreasing in wine consumption of the other person. Preferences of agents A and B are as follows,

$$u_A(x_A, y_A, y_B) = x_A[1 + \max(y_A - y_B, 0)], \quad u_B(x_B, y_B, y_A) = x_B[1 + \max(y_B - y_A, 0)],$$

where A consumes x_A and y_A units of x and y respectively, similarly B's consumption is x_B and y_B. Which of the following is a Pareto optimum allocation.

(a) $x_A = \frac{1}{4}, x_B = \frac{3}{4}, y_A = y_B = \frac{1}{2}$
(b) $x_A = x_B = \frac{1}{2}, y_A = \frac{1}{4}, y_B = \frac{3}{4}$
(c) $x_A = \frac{1}{4}, x_B = \frac{3}{4}, y_A = 1, y_B = 0$
(d) $x_A = \frac{1}{4}, x_B = \frac{3}{4}, y_A = 0, y_B = 1$

A 4.57

(d) $x_A = \frac{1}{4}, x_B = \frac{3}{4}, y_A = 0, y_B = 1$

$((x_A, y_A), (x_B, y_B)) = ((\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{2}))$ is not Pareto optimal because $((x'_A, y'_A), (x'_B, y'_B)) = ((\frac{1}{4}, 0), (\frac{3}{4}, 1))$ is Pareto superior to it because we have $u_A(x_A, y_A, y_B) = u_A(x'_A, y'_A, y'_B)$ and $u_B(x_B, y_B, y_A) < u_B(x'_B, y'_B, y'_A)$. Similarly, $((x_A, y_A), (x_B, y_B)) = ((\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{3}{4}))$ is not Pareto optimal because $((x'_A, y'_A), (x'_B, y'_B)) = ((\frac{1}{2}, 0), (\frac{1}{2}, 1))$ is Pareto superior to it. If we consider $((x_A, y_A), (x_B, y_B)) = ((\frac{1}{4}, 1), (\frac{3}{4}, 0))$, it is not Pareto optimal because $((x'_A, y'_A), (x'_B, y'_B)) = ((\frac{1}{2}, 0), (\frac{1}{2}, 1))$ is Pareto superior to it. So, by elimination, we get $((x_A, y_A), (x_B, y_B)) = ((\frac{1}{4}, 0), (\frac{3}{4}, 1))$ as Pareto optimal.

Alternatively, We can check for the Pareto optimality of $((x_A, y_A), (x_B, y_B)) = ((\frac{1}{4}, 0), (\frac{3}{4}, 1))$ directly. Note that $u_B(x_B, y_B, y_A) = \frac{3}{2}$, so for any feasible allocation $((x'_A, y'_A), (x'_B, y'_B))$ to be Pareto superior to $((\frac{1}{4}, 0), (\frac{3}{4}, 1))$, it must satisfy $y'_B > \frac{1}{2}$. This is because x'_B can not take value greater than 1 and thus, for $u_B(x'_B, y'_B, y'_A) \geq \frac{3}{2}$, we must have $[1 + \max(y'_A - y'_B, 0)] > 1$ and therefore, $y'_B > \frac{1}{2}$. Since, $y'_A < y'_B$, we have $u_A(x'_A, y'_A, y'_B) = x'_A$ and $u_B(x'_B, y'_B, y'_A) = x'_B(1 + y'_B - y'_A)$ and here the best possibility is when $y'_A = 0$. Hence, we get that a possible Pareto superior allocation must satisfy $y'_A = 0$ and $y'_B = 1$ and this observation will therefore tell us the following about the possible Pareto superior allocation: $u_A(x'_A, y'_A, y'_B) = x'_A \geq \frac{1}{4}$ and $u_B(x'_B, y'_B, y'_A) = 2x'_B \geq \frac{3}{2}$ and at least one of the two inequalities must be strict. Adding the two after dividing u_B by 2, we get that such an allocation must satisfy $x'_A + x'_B > \frac{1}{4} + \frac{3}{4} = 1$, violating feasibility. Therefore, $((x_A, y_A), (x_B, y_B)) = ((\frac{1}{4}, 0), (\frac{3}{4}, 1))$ is Pareto optimal.

■

Exercise 4.58 A two-person two-goods pure exchange economy. The initial endowment vectors are $e^1 = (1,0)$ and $e^2 = (0,1)$. The two individuals have identical preferences represented by the utility functions:

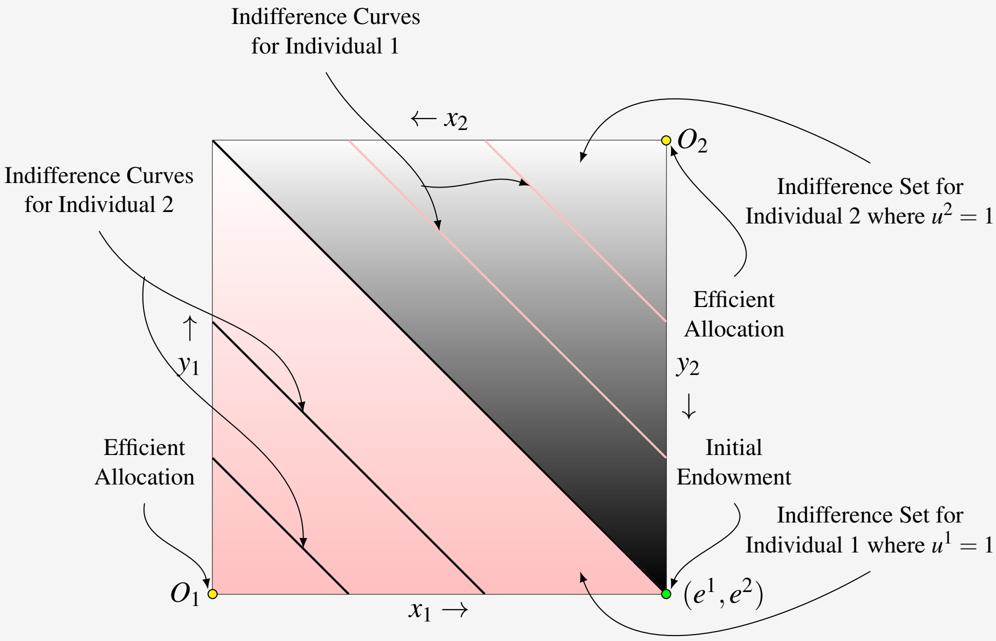
$$u^1(x,y) = u^2(x,y) = \begin{cases} 1 & \text{when } x + y < 1 \\ x + y, & \text{when } x + y \geq 1 \end{cases}$$

where x is the quantity of the first good and y is the quantity of the second good. For this economy the set of Pareto optimum allocations

(a) consists of the entire Edgeworth box
(b) is just the equal division of goods
(c) is a null set
(d) is $\{((0,0),(1,1)),((1,1),(0,0))\}$

A 4.58

(d) is $\{((0,0),(1,1)),((1,1),(0,0))\}$



If we consider the allocation labelled as O_1 i.e. the origin of individual 1, here individual 2 consumes the bundle $(1,1)$ which offers him the highest satisfaction amongst all feasible allocations and there is no other allocation that offers him the same satisfaction, so any change from here will necessarily make individual 2 worse off. Therefore, O_1 is Pareto efficient. Likewise, O_2 is also Pareto efficient. O_1 is Pareto superior to every point in the indifference set of Individual 1 where $u^1 = 1$ (region coloured pink in the box above). Likewise, O_2 is Pareto superior to every point in the indifference set of individual 2 where $u^2 = 1$ (region coloured black in the box above).

Exercise 4.59 A monopolist seller produces a good with constant marginal cost $c \geq 0$. The monopolist sells the entire output to a consumer whose utility from consuming x units of the product is given by $\theta\sqrt{x} - t$, where t is the payment made by the consumer to the monopolist. Suppose, consumer's outside option is 0, i.e., if she does not buy the good from the monopolist, she gets 0 utility. Then, the monopolist's profit is

(a) $\theta/(4c)$
(b) $\theta^2/(4c)$
(c) $c\theta^2$
(d) $c\theta/2$

A 4.59

(b) $\theta^2/(4c)$

Monopolist will solve the following problem

$$\begin{aligned} \max_{(x,t)\geq 0} \quad & t - cx \\ \text{s.t.} \quad & \theta\sqrt{x} - t \geq 0 \end{aligned}$$

Clearly, the above constraint will be binding in optimum i.e. $\theta\sqrt{x} - t = 0$. So, the above problem can be rewritten as

$$\max_{x\geq 0} \quad \theta\sqrt{x} - cx$$

Differentiating the objective, we get FOC as

$$\begin{aligned} \frac{\theta}{2\sqrt{x^*}} - c = 0 \\ \Rightarrow \quad x^* = \frac{\theta^2}{4c^2} \end{aligned}$$

Optimal profit will be

$$\theta\sqrt{x^*} - cx^* = \frac{\theta^2}{2c} - \frac{\theta^2}{4c} = \frac{\theta^2}{4c}$$

Exercise 4.60 Consider an economy consisting of $n \geq 2$ individuals with preference relations defined over the set of alternatives X. Let $S = \{a, b, c, d, e\}$ and $T = \{a, b, c, d\}$ be two subsets of X. Now consider the following statements:

- A. If a is Pareto optimal (PO) with respect to set S, then a is PO with respect to set T.
- B. If a is PO with respect to set T, then a is PO with respect to set S.
- C. If a is PO with respect to set S and b is *not* PO with respect to set T, then a is Pareto superior to b.
- D. If a is the only PO alternative in set S and b is *not* PO with respect to set S, then a is Pareto superior to b.

How many of the above statements are necessarily correct?

(a) 1
(b) 2
(c) 3
(d) 4

A 4.60

(b) 2

Let $\succsim_i$ be the weak preference relation of agent i and $\succ_i$ be his corresponding strict preference relation. We say an alternative x is *Pareto superior* to an alternative y if the following conditions hold:

$$\begin{aligned} & \forall i, \quad x \succsim_i y \\ \& \quad & \exists i, \quad x \succ_i y \end{aligned}$$

We say an alternative x is *Pareto optimal* with respect to set S if there is no alternative in S that is Pareto superior to x. Clearly, statement A is correct because if there is no alternative in S that is Pareto superior to a then there is no alternative in T that is Pareto superior to a since $T \subset S$. Statement B is false. Consider the following example as justification for the claim. Suppose $n = 2$ and the individuals have following preferences over alternatives in S

$$\begin{aligned} & e \succ_1 a \succ_1 b \succ_1 c \succ_1 d \\ & b \succ_2 e \succ_2 a \succ_2 c \succ_2 d \end{aligned}$$

It can be easily seen in the above example that alternative e is the only alternative that is Pareto superior to alternative a. Thus, a is Pareto optimal with respect to set T but it is not Pareto optimal with respect to set S. Statement C is false. The following example shows that it is false. Suppose $n = 2$ and the individuals have following preferences over alternatives in S

$$\begin{aligned} & a \succ_1 c \succ_1 b \succ_1 d \succ_1 e \\ & c \succ_2 b \succ_2 a \succ_2 e \succ_2 d \end{aligned}$$

It can be easily seen in the above example that alternative a is Pareto optimal with respect to S and b is not Pareto optimal with respect to T because c is Pareto superior to b, but a is not Pareto superior to b because individual 2 strictly prefers b over a. Statement D is true. Here is the argument: If alternative a is the only Pareto optimal alternative in set S, then b is not Pareto optimal with respect to set S. This implies that at least one of the alternatives $x \in \{a, c, d, e\}$ is Pareto superior to it. If a is Pareto superior to b, then we are done. If $x \neq a$ is Pareto superior to b, then given that x too is not Pareto optimal implies that there exist $y \notin \{x, b\}$ that is Pareto superior to x. If a is Pareto superior to x, then x being Pareto superior to b implies that a is Pareto superior to b and we are done. If $y \neq a$ is Pareto superior to x then given that y too is not Pareto optimal, we get $z \notin \{x, y, b\}$ that is Pareto superior to y. If a is Pareto superior to y, then y being Pareto superior to x and x to b implies that a is Pareto superior to b and we are done. If $z \neq a$ is Pareto superior to y then given that z too is not Pareto optimal, we get that a is Pareto superior to z and again we establish that a is Pareto superior to b because z is Pareto superior to y, y to x and x to a.